Useful information for the first exam in Math 251:05-10, spring 2006

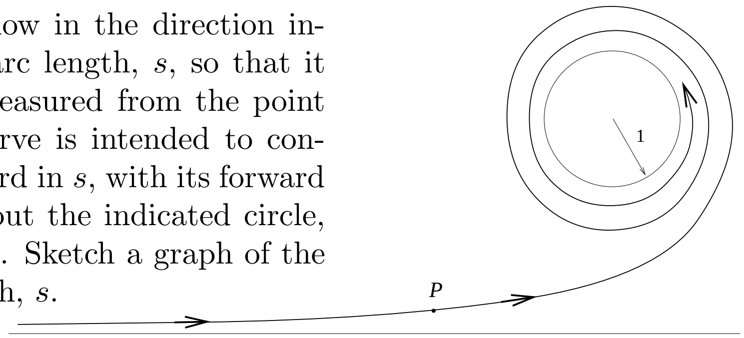
The date of the exam is Friday, February 24. The place and time will be your usual class time and class location. The exam will cover the material of the syllabus up to and including lecture 9 (that is section 14.6 of the text). Here are some exam rules:

• No books or notes. A formula sheet will be handed out with the exam. A *draft version* of the formula sheet will be linked to the course webpage.
• No calculators of any kind may be used during the exam. Please leave answers in "unsimplified" form – so $15^2 + (.07) \cdot (93.7)$ is preferred to 231.559. You should know simple exact values of transcendental functions such as $\cos\left(\frac{\pi}{2}\right)$ and $\exp(0)$. Traditional math constants such as π and e should be left "as is" and not approximated.
• Show your work: an answer alone may not receive full credit.

Here are some problems from past exams. There are almost three times as many problems as a "real" exam would have.

A Sketch two level curves of $f(x,y) = \frac{x-y}{x+y}$. Explain why $\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ does or does not exist.
B Let $f(x,y) = xy^2 + 2x^3$.
a) If one stands on the graph of f at (1,2,6), what is the slope of the graph at that point in its steepest direction?
b) Find $D_{\mathbf{u}}f(1,2)$ if $\mathbf{u}$ is the unit vector in the direction from $(1,2)$ to $(2,4)$.
C Find an equation of the plane containing the points $(1,2,0)$ and $(0,2,1)$ and parallel to the line $x = 1+t$, $y = -1+t$, $z = 2t$.
D If $z = f(x,y)$, $x = 3uv$, and $y = u - v^2$, express $\frac{\partial z}{\partial v}$ in terms of $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, u, and v.
E For the surface defined by $x^5 + y^5 + z^3 + xyz = 0$, find an equation of the tangent plane at $(1,1,-1)$, and the value of $\frac{\partial z}{\partial x}$ at $(1,1,-1)$.
F For the path defined by $\mathbf{f}(t) = \sin 2t\mathbf{i} + t\mathbf{j} + t^3\mathbf{k}$, set up (do not evaluate) an integral for the length of the path from $(0,0,0)$ to $(0,\pi,\pi^3)$. Also, find the angle between $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and the path at $(0,0,0)$.
G and H Let $\mathbf{V} = \langle 1,3,-2\rangle$ and $\mathbf{W} = \langle -1,2,1\rangle$. Find
a) a unit vector in the same direction as $\mathbf{V} + 2\mathbf{W}$.
b) vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ so that $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, $\mathbf{V}_1 \| \mathbf{W}$, and $\mathbf{V}_2 \perp \mathbf{W}$.
c) the area of the parallelogram in $\mathbb{R}^3$ formed by $\mathbf{V}$ and $\mathbf{W}$.
I Use linear approximation at the point $(64,9)$ for an appropriate function of two variables to approximate $\sqrt[3]{63}\sqrt{10}$.
J Find an equation of the tangent plane to the surface $z = x^3y$ at $(1,2,2)$.
K a) If $z = z(x,y)$ is implicitly defined by $xz + \sin(y^2 + z^2) = 1$, find $\frac{\partial z}{\partial x}$.
b) Suppose $f(x,y)$ has continuous partial derivatives and that $x = u+3v$ and $y = v-2u$. If F is defined by $F(u,v) = f(u+3v, v-2u)$ and if it is given that $f_x(4,-1) = 5$, $f_x(1,1) = 8$, $f_y(4,-1) = -3$ and $f_y(1,1) = -6$, find $F_u(1,1)$.
L Find the unit tangent vector $\mathbf{T}$, the principal unit normal vector $\mathbf{N}$, and the curvature κ for the plane curve $\mathbf{r}(t) = (\cos(2t) + 2t\sin(2t))\,\mathbf{i} + (\sin(2t) - 2t\cos(2t))\,\mathbf{j}$ when $t > 0$.
M A particle's acceleration (acceleration is the second derivative of position) is given by $\mathbf{a}(t) = \cos t\mathbf{i} + \sin t\mathbf{J} + 2\mathbf{k}$. Suppose that its position at time 0 is $\langle 1,0,1\rangle$ and its velocity (the first derivative of position) at time 0 is $\langle 0,2,0\rangle$. What is its position at time t?

N A point is moving along the curve below in the direction indicated. Its motion is parameterized by arc length, s, so that it is moving at unit speed. Arc length is measured from the point P (both backward and forward). The curve is intended to continue indefinitely both forward and backward in s, with its forward motion curling more and more tightly about the indicated circle, and, backward, closer to the indicated line. Sketch a graph of the curvature, κ, as a function of the arc length, s.



What are $\lim_{s\to+\infty} \kappa(s)$ and $\lim_{s\to-\infty} \kappa(s)$?
Use complete English sentences to briefly explain the numbers you give.

O and P Suppose the curve C has position vector $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}$.
a) What is the unit tangent vector to C when $t = \pi$?
b) Write an integral for the length of the part of C between $t = 0$ and $t = 2\pi$. DO NOT EVALUATE THE INTEGRAL.
c) Verify that the curvature of this curve is given by the formula $\kappa(t) = \sqrt{\frac{5+4t^2}{(4t^2+1)^3}}$.
d) Even though the first two components of this curve describe uniform circular motion, $\lim_{t\to\infty} \kappa(t) = 0$. Explain briefly why this can happen, using complete English sentences possibly assisted by properly labeled diagrams.

Q a) If $F(x, y) = \frac{3x-4y}{\sqrt{x^2+y^2}}$, briefly explain why $\lim_{(x,y)\to(0,0)} F(x, y)$ does not exist.
b) If $G(x, y) = \frac{3x^2-4y^2}{\sqrt{x^2+y^2}}$, briefly explain why $\lim_{(x,y)\to(0,0)} G(x, y)$ exists.

R Suppose that $G(u, v)$ is a differentiable function of two variables and that $g(x, y) = G\left(\frac{x}{y}, \frac{y}{x}\right)$. Show that $xg_x(x, y) + yg_y(x, y) = 0$.

S a) Find parametric equations for the line through the points $(3, 2, 7)$ and $(-1, 1, 2)$.
b) At what point does this line intersect the plane $x + y + z = 22$?

T Sketch the three contour lines or level curves of the function $f(x, y) = xy^2$ which pass through the points $P = (2, 1)$ and $Q = (-1, 2)$ and $R = (1, 0)$. Label each curve with the appropriate function value and be sure that your drawing is clear and unambiguous.

U Suppose $Q(x, y)$ is defined by the equation $Q(x, y) = e^{xf(y)}$ where f is a differentiable function of one variable with $f(0) = A$, $f'(0) = B$, and $f''(0) = C$. Use this information to compute these quantities: $Q(0, 0)$, $\frac{\partial Q}{\partial x}(0, 0)$, $\frac{\partial Q}{\partial y}(0, 0)$, $\frac{\partial^2 Q}{\partial x^2}(0, 0)$, $\frac{\partial^2 Q}{\partial x\partial y}(0, 0)$, and $\frac{\partial^2 Q}{\partial y^2}(0, 0)$.

V A particle's acceleration (acceleration is the second derivative of position) is given by $\mathbf{a}(t) = \cos t\mathbf{i} + \sin t\mathbf{J} + 2\mathbf{k}$. Suppose that its position at time 0 is $\langle 1, 0, 1\rangle$ and its velocity (the first derivative of position) at time 0 is $\langle 0, 2, 0\rangle$. What is its position at time t.

W and X Let $f(x, y, z) = xy + yz + zx$, and let P be the point $(1, 2, 3)$.
a) Find the direction in which f is increasing most rapidly at P and the corresponding rate of increase.
b) Find one direction in which the directional derivative of F at P is 0.
c) Find the directional derivative of f at P in the direction of $2\mathbf{i} + 4\mathbf{j} - 10\mathbf{k}$.
d) Find equations of the tangent plane and normal line to the surface $f(x, y, z) = 11$ at P.
e) Use the linear approximation to estimate $f(1.1, 1.8.3.2)$.

Y and Z are **free**!