Horseshoes and the Conley Index Spectrum - II: The Theorem is Sharp

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Abstract

Recent work has shown that in the setting of continuous maps on a locally compact metric space the spectrum of the Conley index can be used to conclude that the dynamics of an invariant set is at least as complicated as that of full shift dynamics on two symbols, that is, a horseshoe. In this paper, one considers which spectra are possible and then produce examples which clearly delineate which spectral conditions do or do not allow one to conclude the existence of a horseshoe.

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1 Introduction

In a series of papers [3, 7, 2], ever more general conditions have been provided under which one can prove that an isolated invariant set is semi-conjugate to the full shift dynamics on two symbols. The purpose of this paper is to show that the process cannot be generalized further.

To be more precise, recall that given a topological space X and a continuous function $f : X \to X$, a set $S \subset X$ is *invariant* if $f(S) = S$. For $N \subset X$ its *maximal invariant set* is defined by

$$\mathrm{Inv}(N, f) := \{x \in N \mid \exists \, \{x_n\}_{n=-\infty}^{\infty} \subset N \text{ such that } \; x_{n+1} = f(x_n) \text{ and } x_0 = x\}.$$

A compact set $N \subset X$ is an *isolating neighborhood* if $\mathrm{Inv}(N, f) \subset \mathrm{int}(N)$. An invariant set S is *isolated* if there exists an isolating neighborhood N such that $S = \mathrm{Inv}(N, f)$. Finally, given an isolating neighborhood N a pair of disjoint compact subsets $\{N_0, N_1\}$ forms a *decomposition* of N if $N = N_0 \cup N_1$. Throughout this paper, given such a decomposition, let $S := \mathrm{Inv}(N, f)$ and $S_i := \mathrm{Inv}(N_i, f)$ for $i = 0, 1$.

The above mentioned theorems of [3, 7, 2] all take the following form. Given an isolating neighborhood N with decomposition $\{N_0, N_1\}$ and the appropriate cohomological information there exist a continuous surjective map

$$\rho : S \to \Sigma_2^+ := \{a = (a_n)_{n=0}^{\infty} \mid a_n \in \{0, 1\}\}$$

and a positive integer d such that

$$T \circ \rho = \rho \circ f^d,$$

where $T : \Sigma_2^+ \to \Sigma_2^+$ is the shift map

$$(T(a))_n = a_{n+1}.$$

Furthermore, if $x \in \rho^{-1}(a)$ then $f^{dn}(x) \in N_{a_n}$. In other words, the decomposition provides the sets upon which the symbolic dynamics is defined.

The cohomological information is obtained via the Conley index which associates to each isolated invariant set S a graded vector space $CH^*(S)$ and a graded linear automorphism $\chi^*(S) : CH^*(S) \to CH^*(S)$. The focus of this paper is on determining the possible structure of the dynamics on S, given the spectrum of $\chi^*(S)$ and $\chi^*(S_i)$, $i = 0, 1$. For this, we will make use of the following notation. Given a linear operator M, let $\sigma(M)$ denote, the set of eigenvalues of M where the eigenvalues are repeated according to their multiplicity. Given two sets A and B, their amalgamation $A \amalg B$ is obtained by taking the union of A and B, but by treating elements common to both A and B as distinct elements in $A \amalg B$.

Finally, two types of sets play an important role in our analysis. A set of complex numbers Q is a *power* set if there exist $x \in \mathbf{Z}$ and $n \in \mathbf{Z}^+$, $n > 1$, such that

$$Q = \{z \in \mathbf{C} \mid z^n = x\}.$$

A set of complex numbers Q is *cyclic* if for some $I \in \mathbf{Z}^+$, $Q = \cup_{i=1}^{I} Q_i$, where Q_i is a power set, for all $i = 1, \ldots, I$, and for all $i \neq j$, $Q_i \cap Q_j = \emptyset$.

The set of the periods of a cyclic set Q is

$$\mathcal{E} = \mathcal{E}_Q := \{n_i : i = 1, \ldots, I\}.$$

A set P is *admissible*, if it represents the collection of *non-zero* roots, counting multiplicity, of an integral polynomial of the form

$$x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1 x^1 + a_0,$$

where $a_j \in \mathbf{Z}$, $j = 0, \ldots, m-1$.

The most general result involving the Conley index and the existence of a semi-conjugacy onto the full two shift is the following.

Theorem 1.1 [2, Theorem 1.1] *Let $f : X \to X$ be a continuous function on a locally compact metric space. Assume that $\{N_0, N_1\}$ form a decomposition of an isolating neighborhood $N \subset X$. If for some positive integer q*

$$\sigma(\chi^q(S_0)) \amalg \sigma(\chi^q(S_1)) \not\subset \sigma(\chi^q(S))$$

or

$$\sigma(\chi^q(S)) \setminus (\sigma(\chi^q(S_0)) \amalg \sigma(\chi^q(S_1)))$$

is not cyclic, then there exist a positive integer d and a continuous surjection $\rho : S \to \Sigma_2^+$ such that

$$T \circ \rho = \rho \circ f^d.$$

One of the objectives of this paper is to show that this theorem is sharp. More precisely, given any other assumption concerning the relationship between the spectra, there exist examples for which $S \setminus (S_0 \cup S_1)$ consists of only a finite number of orbits.

¿From Theorem 1.1 there are only two possibilities which need to be considered

$$\sigma(\chi^q(S_0)) \amalg \sigma(\chi^q(S_1)) = \sigma(\chi^q(S)) \tag{1}$$

or $\sigma(\chi^q(S_0)) \amalg \sigma(\chi^q(S_1))$ is strictly contained in $\sigma(\chi^q(S))$ and

$$\Sigma := \sigma(\chi^q(S)) \setminus (\sigma(\chi^q(S_0)) \amalg \sigma(\chi^q(S_1))) \tag{2}$$

is cyclic.

To fully understand the implications of these conditions we need (a) to know what constraints exist on the spectra and (b) to have a general procedure by which we can take spectral data and produce examples. The latter point is the heart of Section 4.

With regard to (a), as will be explained further in Section 2, for any isolated invariant set S, $\sigma(\chi^*(S))$ consists of the non-zero spectrum of a cohomology map. Therefore, we have the following result.

Proposition 1.2 *For any isolated invariant set S and for any $n \in \mathbf{Z}$, $\sigma(\chi^n(S))$ is an admissible set.*

Observe that it is possible for $\sigma(\chi^n(S)) = \emptyset$. In Section 5, we shall prove a converse result.

Theorem 1.3 *Given any admissible set P and a positive integer q, there exist a set $X \subset \mathbf{R}^{q+2}$, a C^∞-diffeomorphism $f : X \to f(X) \subset X$ and an isolating neighborhood $N \subset X$ such that $S = \mathrm{Inv}(N, f)$ is hyperbolic and*

$$\sigma(\chi^n(S)) = \begin{cases} P & \text{if } n = q, \\ \emptyset & \text{otherwise.} \end{cases}$$

Our interest, however, is in the relationship between $\sigma(\chi^*(S))$ and $\sigma(\chi^*(S_i))$, $i = 0, 1$. Again, given the description of $\chi^*(S)$ in Section 2, the following is clear.

Proposition 1.4 *Let $f : X \to X$ be a continuous function on a locally compact metric space. Assume $N \subset X$ is an isolating neighborhood with decomposition N_0 and N_1. Then for all $n \in \mathbf{Z}^+$,*

$$\mathrm{tr}(\chi^n(S_0)) + \mathrm{tr}(\chi^n(S_1)) = \mathrm{tr}(\chi^n(S)).$$

It remains an open question whether this is the only constraint on the spectra. In other words, given admissible sets P and P_i, $i = 0, 1$ satisfying the relation

$$\sum_{\lambda \in P_0 \amalg P_1} \lambda = \sum_{\lambda \in P} \lambda$$

do there exist a locally compact metric space X, a continuous function $f : X \to X$, and an isolating neighborhood $N \subset X$ with decomposition $\{N_0, N_1\}$ such that

$$\sigma(\chi^*(S)) = P \quad \text{and} \quad \sigma(\chi^*(S_i)) = P_i, \; i = 0, 1 \; ?$$

The following is a partial result in this direction. Observe that Proposition 1.4 implies: if $\sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1)) \subset \sigma(\chi^n(S))$, then

$$\sum_{\lambda \in \Sigma} \lambda = 0,$$

where $\Sigma := \sigma(\chi^n(S)) \setminus (\sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1)))$.

Proposition 1.5 *Let P, P_0 and P_1 be admissible sets such that*

$$P_0 \amalg P_1 \subset P \quad \textit{and} \quad \sum_{\lambda \in P \setminus (P_0 \amalg P_1)} \lambda = 0.$$

Then, given a positive integer q, there exist a set $X \subset \mathbf{R}^{q+2}$, a C^∞-diffeomorphism $f : X \to f(X) \subset X$ and an isolating neighborhood $N \subset X$ with decomposition N_0 and N_1 such that $S = \mathrm{Inv}(N, f)$ is hyperbolic,

$$\sigma(\chi^n(S)) = \begin{cases} P & \textit{if } n = q, \\ \emptyset & \textit{otherwise} \end{cases} \quad \textit{and} \quad \sigma(\chi^n(S_i)) = \begin{cases} P_i & \textit{if } n = q, \\ \emptyset & \textit{otherwise.} \end{cases}$$

With these results as background, we return to the spectral conditions of (1) and (2). To understand the implications of (1), we need the following concept. The decomposition (N_0, N_1) of N is an *attractor-repeller decomposition* if N_0 is positively invariant in N under f, i.e., given $x \in N_0$ if $f(x) \in N$ then $f(x) \in N_0$. Given an attractor-repeller decomposition of N, (S_0, S_1) forms an attractor-repeller decomposition for S. Observe that by definition, if $x \in S$, then there exists a sequence $\{x_n\} \subset S$ such that $x_{n+1} = f(x_n)$ and $x_0 = x$. In the setting of an attractor-repeller pair, if $x \in S$, then either $x \in S_0 \cup S_1$ or $\lim_{n\to\infty} \mathrm{dist}(x_n, S_0) = 0$ and $\lim_{n \to -\infty} \mathrm{dist}(x_n, S_1) = 0$, where $x_0 := x$ and $x_{n+1} := f(x_n)$.

The relationship between the spectra of the Conley index for attractor-repeller pairs and that of the entire invariant set is fairly simple.

Theorem 1.6 [3, Theorem 2.2] *Let (N_0, N_1) be an attractor-repeller decomposition of N. Then,*

$$\sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1)) = \sigma(\chi^n(S)),$$

for all $n \in \mathbf{Z}$.

We are interested in understanding to what extent the converse of this theorem holds. The following result follows directly from Theorem 1.3.

Theorem 1.7 *Given admissible sets P_0 and P_1 and a positive integer q, there exist a set $X \subset \mathbf{R}^{q+2}$, a C^∞-diffeomorphism $f : X \to f(X) \subset X$, and an isolating neighborhood $N \subset X$ with an attractor-repeller decomposition (N_0, N_1) such that S is hyperbolic,*

$$\sigma(\chi^q(S_i)) = P_i, \quad i = 0, 1,$$

and for all $n \in \mathbf{Z}^+$,

$$\sigma(\chi^n(S)) = \sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1)).$$

Turning now to (2) we shall prove the following result.

Theorem 1.8 *Let P and P_i, $i = 0, 1$ be admissible sets such that $\Sigma := P \setminus (P_0 \amalg P_1)$ is cyclic and let q be a positive integer. Then, there exist a set $X \subset \mathbf{R}^{q+2}$, a C^∞-diffeomorphism $f : X \to f(X) \subset X$, and an isolating neighborhood $N \subset X$ with a decomposition $\{N_0, N_1\}$ such that S is hyperbolic,*

$$\sigma(\chi^n(S_i)) = \begin{cases} P_i & \text{if } n = q, \\ \emptyset & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma(\chi^n(S)) = \begin{cases} P & \text{if } n = q, \\ \emptyset & \text{otherwise,} \end{cases}$$

and $S \setminus (S_0 \cup S_1)$ consists of a finite collection of periodic orbits with minimal period k, where $k \in \mathcal{E}_\Sigma$.

Notice that Theorem 1.7 and Theorem 1.8 demonstrate Theorem 1.1 is sharp. Moreover, the conclusion of Theorem 1.8 applies to continuous maps for which (2) is cyclic. This shows that we are st ill able to determine a non-trivial lower bound for the dynamics. The proof makes use of the Lefschetz number (to be defined in Section 2) which requires an additional constraint on the space X.

In the definition of cyclic set, the uniqueness of the decomposition was not required. In the next result, we will also impose an extra condition on a cyclic set. A cyclic set Q which is of the form $Q = \cup_{i=1}^{I} Q_i$, for some $I \in \mathbf{Z}^+$ and Q_i a power set, is called *irreducible* if for any $(i_1, \ldots, i_{I-1}) \in \{1, \ldots, I\}^{I-1}$, $\cup_{s=1}^{I-1} Q_{i_s}$ is not a power set.

Theorem 1.9 *Let $f : X \to X$ be a continuous function on a locally compact ENR metric space. Assume that $\{N_0, N_1\}$ form a decomposition of an isolating neighborhood $N \subset X$. If for some positive integer q, $\sigma(\chi^n(S)) = \sigma(\chi^n(S_0)) = \sigma(\chi^n(S_1)) = \emptyset$ for $n \neq q$ and*

$$\Sigma := \sigma(\chi^q(S)) \setminus (\sigma(\chi^q(S_0)) \amalg \sigma(\chi^q(S_1)))$$

is an irreducible cyclic set. Then either (*i*) *there exist a positive integer* d *and a continuous surjection* $\rho : S \to \Sigma_2^+$ *such that*

$$T \circ \rho = \rho \circ f^d$$

or

(*ii*) $S\setminus(S_0 \cup S_1)$ *consists of a finite collection of periodic orbits with minimal period* k*, where* $k \in \mathcal{E}_\Sigma$.

Except for Theorem 1.9, the results stated above take the form of examples in which index information permits simple symbolic dynamics with respect to the decomposition $\{N_0, N_1\}$. However, it is important that the reader not be lulled into believing that the index information can be used to obtain upper bounds on the complexity of the dynamics on S. To emphasize this point we state the following proposition which will be proven in Section 6.

Proposition 1.10 *Let* P, P_0 *and* P_1 *be admissible sets such that*

$$P_0 \amalg P_1 \subset P \quad \text{and} \quad \sum_{\lambda \in P\setminus(P_0 \amalg P_1)} \lambda = 0.$$

Then, given a positive integer q *there exist a set* $X \subset \mathbf{R}^{q+2}$*, a* C^∞*-diffeomorphism* $f : X \to f(X) \subset X$ *and an isolating neighborhood* $N \subset X$ *such that* $S = \mathrm{Inv}(N, f)$ *is hyperbolic,*

$$\sigma(\chi^n(S)) = \begin{cases} P & \text{if } n = q, \\ \emptyset & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma(\chi^n(S_i)) = \begin{cases} P_i & \text{if } n = q, \\ \emptyset & \text{otherwise.} \end{cases}$$

Furthermore, there exist an isolated invariant subset $S' \subset S$ *and a continuous surjective map* $\rho : S' \to \Sigma_2^+$ *such that restricted to* S'

$$T \circ \rho = \rho \circ f^2.$$

The outline of this paper is as follows. Section 2 provides a brief review of the Conley index theory. Section 3 is devoted to the proof of Theorem 1.9 and Section 4 contains the key result to define the examples. The proof is constructive. In Section 5, this construction is then modified in appropriate ways to prove Proposition 1.5, Theorems 1.3, 1.7 and 1.8. Finally, the last section contains the proof of Proposition 1.10.

2 The Conley Index

The characterization of the Conley index used in the results stated in the Introduction is due to M. Mrozek [4] and is defined as follows. An *index pair* for an isolated invariant set S with respect to a continuous map f consists of a pair of compact sets (N, L) satisfying:

1. $S = \mathrm{Inv}(\mathrm{cl}(N \setminus L), f) \subset \mathrm{int}(N \setminus L)$,
2. $f(L) \cap N \subset L$,
3. $f(N \setminus L) \subset N$.

Observe that $f : (N, L) \to (N \cup f(L), L \cup f(L))$ and that excision gives rise to an isomorphism $e^* : H^*(N, L) \to H^*(N \cup f(L), L \cup f(L))$. Thus, we can define $F^*_{N,L} := f^* \circ e^* : H^*(N, L) \to H^*(N, L)$. Let

$$\mathrm{gker} F^*_{N,L} := \{v \in H^*(N, L) \mid (F^*_{N,L})^k(v) = 0 \text{ for some } k \in \mathbf{Z}^+\}.$$

Mrozek showed that $CH^*(S) := H^*(N, L)/\mathrm{gker} F^*_{N,L}$ and the induced automorphism $\chi^*(S) : CH^*(S) \to CH^*(S)$ are invariants of S, i.e. they are independent of the index pair (N, L) of S. Observe that $F^*_{N,L}$ can be represented as a matrix with integer entries; from which Theorem 1.2 follows.

The problem with using Mrozek's approach for the proof is that it is extremely difficult to compare the spectra of the indices of different isolated invariant sets. For this reason, we need to turn to an extension of the Conley index given by Szymczak [8]. An outline of his approach is given below but the reader is referred to [8] for further details.

Given an index pair (N, L), observe that f induces a continuous map on the pointed quotient spaces

$$f_{N,L} : (N/L, [L]) \to (N/L, [L]).$$

The index pair (N, L) is called *regular* if $f^{-1}_{N,L}([L])$ is a neighborhood of $[L]$. By [6, Corollary 2] regular index pairs always exist.

Let S be an isolated invariant set with respect to a continuous map $f : X \to X$. The *Conley index* of S is the class of all objects isomorphic to $\left(H^*(N/L, [L]), f^*_{N,L}\right)$ in $\mathcal{V}^*_m$, where $\mathcal{V}^*$ is the category of graded vector spaces with graded linear maps. In [8], it is shown that this definition is independent of the index pair chosen and hence the index can be written as

$$h^*(S, f) := \left[\left(H^*(N/L, [L]), f^*_{N,L}\right)\right].$$

In what follows, $h^*(S,f)=0$ will denote the equivalence class containing the trivial graded vector space with the trivial graded map.

An immediate and fundamental consequence of above definition is

Theorem 2.1 *If $h^*(S,f)\neq 0$, then $S\neq\emptyset$.*

The notation that will be introduced next is used to relate the indices of S with those of isolated invariant subsets of S. Given $\omega=(\omega_0,\omega_1,\ldots,\omega_k)\in\{0,1\}^{k+1}$, define

$$N_\omega := \bigcap_{j=0}^{k} f^{-j}(N_{\omega_j})$$

and

$$S_\omega := \mathrm{Inv}(N_\omega, f^{k+1}).$$

Let (K,L) be a regular index pair for S such that $K\subset N$. Let $\Pi : K\to K/L$ be the quotient map and given $x\in K$ set $[x]:=\Pi(x)$. To simplify the notation set $(Y,\bar{y}) := (K/L,[L])$ and $F:=f_{K,L} : (Y,\bar{y})\to(Y,\bar{y})$. Define

$$Y_i := \Pi(K\cap N_i)\cup\{\bar{y}\},\quad i=0,1,$$

and continuous maps $r_i, F_i : (Y,\bar{y})\to(Y,\bar{y})$ given by

$$r_i([x]) := \begin{cases} x & \text{if } x\in Y_i, \\ \bar{y} & \text{otherwise}\end{cases}$$

and

$$F_i := F\circ r_i.$$

Given $\omega\in\{0,1\}^{k+1}$ set

$$F_\omega := F_{\omega_k}\circ\ldots\circ F_{\omega_1}\circ F_{\omega_0}.$$

The following proposition indicates that the Conley indices of isolated invariant subsets of S can be viewed in terms of different graded linear maps defined on the same graded vector space.

Proposition 2.2 [7, Lemma 3.1] *Given $\omega\in\{0,1\}^{k+1}$*

$$h^*(S_\omega, f^{k+1}) = [(H^*(Y,\bar{y}), F^*_\omega)].$$

The next result shows that the spectral properties of F_i^* agree with the spectral information assumed in the statement of the results in the Introduction.

Proposition 2.3 [2, Proposition 2.3] *The spectral decompositions of isomorphic objects in $\mathcal{V}^*_m$ differ only in the space corresponding to the zero eigenvalue.*

We finish this section with the following definitions. First, observe that given any linear map $M : V \to V$, there exists an integer I such that for any $n \geq I$,

$$\operatorname{rank}(M^I) = \operatorname{rank}(M^n) \in \mathbf{Z}^+ \cup \{\infty\}.$$

The number

$$\operatorname{arank}(M) := \operatorname{rank}(M^I)$$

is called *asymptotic rank* of M.

An object (V^*, M^*) in $\mathcal{V}_m^*$ is of finite type if there exists $(\overline{V}^*, \overline{M}^*)$ in $\mathcal{V}_m^*$ equivalent to (V^*, M^*) such that $\dim \overline{V}^q$ is finite for all $q \in \mathbf{Z}$ and zero except for a finite number of $q \in \mathbf{Z}$, and $\overline{M}^q$ is an isomorphism.

If (V^*, M^*) is of finite type, we define its *Lefschetz number* by

$$\Lambda(V^*, M^*) = \sum_{q \in \mathbf{Z}} (-1)^q \operatorname{tr}(\overline{M}^q).$$

In the proof of Theorem 1.9, we need to be able to compute the Lefschetz number of the Conley index. Therefore the question that arises is given S, an isolated invariant set for a continuous map f defined on a locally compact metric space X, $h^*(S, f)$ is of finite type? M. Mrozek [5] showed that for an ANR metric space $(CH^*(S), \chi^*(S))$ is of finite type. Since we are also interested in determining fixed points via index information we will follow A. Szymczak and assume X is an ENR space. For proofs of the results, we refer [7, Lemma 7.2].

3 Proof of Theorem 1.9

We start briefly reviewing the notation introduced in [2] which was used to generate and discuss products of matrices. Let $\omega = (\omega_0, \ldots, \omega_k) \in \{0,1\}^{k+1}$. Its *length* is denoted by $p(\omega) = k + 1$. The sequence obtained by repeating ω i-times is denoted by $[\omega]^i \in \{0,1\}^{(k+1)i}$, i.e.

$$[\omega]^i := (\omega_0, \ldots, \omega_k, \omega_0, \ldots, \omega_k, \ldots, \omega_0, \ldots, \omega_k).$$

The sequence ω is *minimal*, if there does not exist $\alpha \in \{0,1\}^j$ such that $\omega = [\alpha]^i$, for some i. Now let $A_i : V \to V$, $i = 0, 1$, be two linear maps defined on a vector space V. Given $\omega \in \{0,1\}^{k+1}$ let

$$A(\omega) := A_{\omega_0} \circ A_{\omega_1} \circ \ldots \circ A_{\omega_k}.$$

Finally, set

$$\mathcal{G} = \mathcal{G}_{\{A_0, A_1\}} := \{\omega \in \{0,1\}^{n+1} \mid n \in \mathbf{N},\ \omega \text{ is minimal},\ A(\omega) \text{ is not nilpotent}\}.$$

The symbol $\#(\mathcal{G})$ will be used to denote the cardinality of the set $\mathcal{G}$.

We recall a couple of results proven in [2]. The first one tells us that if $\#(\mathcal{G}) = \infty$, we can determine generators of a free semigroup.

Proposition 3.1 [2, Proposition 3.4] *Let $A_0, A_1 : V \to V$ be linear maps such that $\#(\mathcal{G}) = \infty$. Then, there exist a positive integer d and distinct sequences $\alpha, \beta \in \{0,1\}^{d+1}$ with the following property. Let*

$$\begin{aligned} \Psi_0 &= A(\alpha) \\ \Psi_1 &= A(\beta), \end{aligned}$$

then given any positive integer n and any sequence $\omega \in \{0,1\}^n$, $\Psi(\omega)$ is not nilpotent.

The next result gives a polynomial relation between the nonzero spectrum of $A_0 + A_1$, A_0 and A_1 provided $\#(\mathcal{G})$ is finite.

Proposition 3.2 [2, Proposition 3.1] *If $\#(\mathcal{G}) < \infty$, then for all $z \in \mathbf{C}$,*

$$\det(I - zA) = \prod_{\omega \in \mathcal{G}} \det\left(I - z^{p(\omega)} A(\omega)\right)^{1/p(\omega)}. \tag{3}$$

The following proposition provides for the existence of the non-nilpotent matrices which will give the desired periodic points. Given a linear map M, let $\sigma^{\times}(M)$ denote the nonzero spectrum of M.

Proposition 3.3 *Let $A_0, A_1 : V \to V$ be linear maps such that*

$$\sigma^{\times}(A_0) \amalg \sigma^{\times}(A_1) \subset \sigma^{\times}(A_0 + A_1),$$

$$\Sigma := \sigma^{\times}(A_0 + A_1) \setminus (\sigma^{\times}(A_0) \amalg \sigma^{\times}(A_1))$$

is cyclic, and $\#(\mathcal{G})$ is finite. If $k \in \mathcal{E}_{\Sigma}$ then there exists $\omega \in \{0,1\}^k$ such that $A(\omega)$ is not nilpotent.

Proof. Since Σ is a cyclic set, there exist $s \in \mathbf{Z}^+$, $r_i \in \mathbf{Z}^+$, $r_i > 1$ for $i = 1, \ldots, s$, and $\nu_i \in \mathbf{Z}$ such that

$$\Sigma := \amalg_{i=1}^{s} \{\nu_{ij} \mid \nu_{ij}^{r_i} = \nu_i, j = 1, \ldots, r_i\}.$$

¿From Proposition 3.2, it follows that

$$\det(I - zA) = \prod_{\omega \in \mathcal{G}} \det\left(I - z^{p(\omega)} A(\omega)\right)^{1/p(\omega)}.$$

The hypothesis implies that $\mathcal{G}^* := \mathcal{G} \setminus \{0,1\}$ is non-empty. Therefore the previous equality can be written as follows:

$$\prod_{i=1}^{s}\prod_{j=1}^{r_i}(1 - z\nu_{ij}) = \prod_{\omega\in\mathcal{G}^*} \det\left(I - z^{p(\omega)}A(\omega)\right)^{1/p(\omega)}.$$

Since $\nu_{ij}^{r_i} = \nu_i$, we finally get

$$\prod_{i=1}^{s}(1 - z^{r_i}\nu_i) = \prod_{\omega\in\mathcal{G}^*} \det\left(I - z^{p(\omega)}A(\omega)\right)^{1/p(\omega)}. \tag{4}$$

In order to have a better understanding of the above equality, the right-hand side of equation (4) shall be written in terms of the eigenvalues of $A(\omega)$. To do this, some notation needs to be introduced. Let $\mathcal{G}^* := \{\omega_1, \dots, \omega_k\}$, for some positive integer k. For $\omega_t \in \mathcal{G}^*$, let

$$\sigma^{\times}(A(\omega_t)) = \{\lambda_1(\omega_t), \dots, \lambda_{q(\omega_t)}(\omega_t)\},$$

where $q(\omega_t)$ denotes the cardinality of the multiset $\sigma^{\times}(A(\omega_t))$. Therefore, equation (4) becomes

$$\prod_{i=1}^{s}(1 - z^{r_i}\nu_i) = \prod_{t=1}^{k}\prod_{l=1}^{q(\omega_t)}\left(1 - z^{p(\omega_t)}\lambda_l(\omega_t)\right)^{1/p(\omega_t)}.$$

Define $p = \prod_{t=1}^{k} p(\omega_t)$ and take the p^{th} power of both sides of the equality above. This leads to

$$\prod_{i=1}^{s}(1 - z^{r_i}\nu_i)^p = \prod_{t=1}^{k}\prod_{l=1}^{q(\omega_t)}\left(1 - z^{p(\omega_t)}\lambda_l(\omega_t)\right)^{p_t}, \tag{5}$$

where $p_t = \prod_{j\neq t} p(\omega_j)$. Since the roots of the polynomial in the left-hand side of (5) are $1/\nu_{ij}$, we have

$$\prod_{i=1}^{s}(1 - z^{r_i}\nu_i)^p = \prod_{t=1}^{k}\prod_{i=1}^{s}\prod_{j=1}^{r_i}\left(1 - z^{p(\omega_t)}\nu_{ij}^{p(\omega_t)}\right)^{p_t m_{t_{ij}}}, \tag{6}$$

where $m_{t_{ij}}$ is a non-negative integer. To finish, one needs to verify which restrictions the above inequality imposes on k, $p(\omega_t)$ and $m_{t_{ij}}$. Notice that if there exists t_0 such that $p(\omega_{t_0}) \neq r_i$ for all i, then $(1/\nu_{ij})^{1/p(\omega_{t_0})}$ is a root of the polynomial in the left-hand side of (6), but this is a contradiction. Therefore, $p(\omega_t) = r_{i_t}$. This fact implies that

$$m_{t_{ij}} = \begin{cases} 1 & \text{if } i = i_t, \\ 0 & \text{otherwise} \end{cases}$$

and $k = s$. Hence $\mathcal{G}^* = \{\omega_1, \ldots, \omega_s\}$, where $\omega_i \in \{0,1\}^{r_i}$. By the definition of $\mathcal{G}$, $A(\omega_i)$ is not nilpotent. ∎

Remark 3.4 Notice that the sequences $\omega_i \in \{0,1\}^{r_i}$ found have nonzero spectrum $\sigma^\times(A(\omega_i))$ equal to $\{\nu_i\}$.

Proof of Theorem 1.9. Let $F_i : (Y, \bar{y}) \to (Y, \bar{y})$ be as defined in Section 2 and consider

$$A_i := F_i^n : H^n(Y, \bar{y}) \to H^n(Y, \bar{y}), \quad i = 0, 1.$$

Then $A_0 + A_1 := F^n : H^n(Y, \bar{y}) \to H^n(Y, \bar{y})$. By Proposition 2.3,

$$\sigma^\times(A_i) = \sigma(\chi^n(S_i)) \quad i = 0, 1,$$

and

$$\sigma^\times(A_0 + A_1) = \sigma(\chi^n(S)).$$

Suppose $\#(\mathcal{G}_{\{A_0, A_1\}}) = \infty$. In this case, choose a positive integer d and distinct sequences $\alpha, \beta \in \{0,1\}^{d+1}$ according to Proposition 3.1. Let $\Psi_0 := A(\alpha)$ and $\Psi_1 := A(\beta)$. The proof of the existence of a semi-conjugacy is exactly as the proof of Theorem 1.1 done in [2].

The next possibility is the case that $\#(\mathcal{G}_{\{A_0, A_1\}})$ is finite. Since Σ is an irreducible cyclic set, given $k \in \mathcal{E}_\Sigma$ there exists $\omega \in \{0,1\}^k$ satisfying the conclusions of Proposition 3.3. By Proposition 2.2,

$$h^*(S_\omega, f^k) = [(H^*(Y, \bar{y}), A(\omega))].$$

Since X is *ENR*, it follows $[(H^*(Y, \bar{y}), A(\omega))]$ is of finite type. By hypothesis,

$$\Lambda(h^*(S_\omega, f^k)) = (-1)^q \mathrm{tr}(\chi^q(S_\omega)) = (-1)^q \mathrm{tr}(A(\omega)),$$

and Proposition 3.3 implies that

$$\Lambda(h^*(S_\omega, f^k)) \neq 0.$$

Therefore, by [7, Lemma 7.2], k has a fixed point. Notice that the fixed point of f^k in S_ω is a periodic point of f. The final step of the proof is to show that this periodic point has principal period equal to k. This follows as in [7, Theorem 2.3]. ∎

4 Spectral data and dynamics

The next result shows how algebra can be realized in dynamics. Its proof is constructive and it will be used to generate the set of examples needed in the proofs of Propositions 1.5 and 1.10 and Theorems 1.3, 1.7 and 1.8.

Proposition 4.1 *Let M be an $n \times n$ matrix over $\mathbf{Z}$. Given a positive integer q, there exist a set $X \subset \mathbf{R}^{q+2}$, a C^∞-diffeomorphism $f : X \to f(X) \subset X$ and an isolating neighborhood $N \subset X$ such that $S = \mathrm{Inv}(N, f)$ is hyperbolic and*

$$\sigma(\chi^n(S)) = \begin{cases} \sigma^\times(M) & \text{if } n{=}q, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. The first step is to prove the result for $q = 1$. It is, then, a trivial matter to obtain the general result.

Unfortunately, the construction requires the introduction of a considerable amount of notation. Let the entries of the matrix M be denoted by $a_{i,j} \in \mathbf{Z}$, $i, j = 1, \dots, n$. Define

$$X := \{(x, y, z) \in \mathbf{R}^3 \mid 0 \le x \le 1,\ -1 \le y \le 1,\ 0 \le z \le 2n + 1\}.$$

For $j = 1, \dots, n$, consider disjoint unit cubes

$$D^j := \{(x, y, z) \in \mathbf{R}^3 \mid 0 \le x \le 1,\ -1 \le y \le 1,\ 2j - 1 \le z \le 2j\}.$$

Given $i, j = 1, \dots, n$, define

$$\mathcal{J}^i := \{k \mid a_{i,k} \ne 0\} \quad \text{and} \quad \mathcal{I}^j := \{k \mid a_{k,j} \ne 0\}.$$

Fix j. If $\mathcal{I}^j \ne \emptyset$, define a set of numbers

$$\mathcal{Z}^j := \{z^{i,j}_{k,1}, z^{i,j}_{k,2} \mid i \in \mathcal{I}^j, k = 1, \dots, |a_{i,j}|\} \subset (2j - 1, 2j)$$

with the following properties

$$z^{i,j}_{k,1} < z^{i,j}_{k,2} < z^{i,j}_{k+1,1} < z^{i+1,j}_{v,1},$$

for all $v = 2, \dots, |a_{s+1,j}|$. Let $i^j_{\mathrm{bot}} := \min\{i \mid a_{i,j} \ne 0\}$ and $i^j_{\mathrm{top}} := \max\{i \mid a_{i,j} \ne 0\}$. Observe that $z^{i^j_{\mathrm{bot}},j}_{1,1}$ and $z^{i^j_{\mathrm{top}},j}_{|a_{i^j_{\mathrm{top}},j}|,2}$ are the smallest and largest values in $\mathcal{Z}^j$. For

$i \in \mathcal{I}^j$ and $i < i^j_{\mathrm{top}}$, let $l^j_i := \min\{p : p > i \text{ and } a_{p,j} \neq 0\}$. Using these numbers, D^j can be partitioned into horizontal slabs. Let

$$\begin{aligned}
H^{i,j}_k &:= \{(x,y,z) \in D^j \mid z^{i,j}_{k,1} \leq z \leq z^{i,j}_{k,2}\},\\
E^{i,j}_k &:= \{(x,y,z) \in D^j \mid z^{i,j}_{k,2} \leq z \leq z^{i,j}_{k+1,1}\},\\
G^{i,j} &:= \{(x,y,z) \in D^j \mid z^{i,j}_{|a_{i,j}|,2} \leq z \leq z^{l^j_i,j}_{1,1}\},\\
A^j_{\mathrm{bot}} &:= \{(x,y,z) \in D^j \mid 2j-1 \leq z \leq z^{i^j_{\mathrm{bot}},j}_{1,1}\},\\
A^j_{\mathrm{top}} &:= \{(x,y,z) \in D^j \mid z^{i^j_{\mathrm{top}},j}_{|a_{i^j_{\mathrm{top}},j}|,2} \leq z \leq 2j\}.
\end{aligned}$$

Observe that A^j_{bot} and A^j_{top} are the bottom and the top slabs of D^j.

Similarly, for fixed j, if $\mathcal{J}^j \neq \emptyset$, define vertical slabs

$$V^{i,j}_k := \{(x,y,z) \in D^j \mid y^{i,j}_{k,1} \leq y \leq y^{i,j}_{k,2}\},$$

where $i \in \mathcal{J}^j$, $k = 1, \ldots, |a_{j,i}|$ and

$$0 < y^{u,j}_{k,1} < y^{u,j}_{k,2} < y^{u,j}_{k+1,1} < y^{u+1,j}_{v,1} < 1,$$

for all $u = 1, \ldots, n-1$ and $v = 2, \ldots, |a_{j,s+1}|$.

Let $\mu \geq 0$ be a constant which is chosen large enough in order to guarantee the sets constructed below do not intersect. Consider the following affine map, $S : \mathbf{R} \to \mathbf{R}$ given by

$$S(x) := \frac{x}{2\mu} + \frac{1}{2\mu} + \frac{1}{2}.$$

Six more affine maps are also defined. Let $L^{i,j}_{k,+}, L^{i,j}_{k,-}, T^{j,\mathrm{bot}}_{+}, T^{j,\mathrm{bot}}_{-}, T^{j,\mathrm{top}}_{+}, T^{j,\mathrm{top}}_{-} : \mathbf{R}^3 \to \mathbf{R}^3$ be given by

$$L^{i,j}_{k,+}(x,y,z) := \left(S(x),\ \frac{1}{2}(y^{j,i}_{k,2} - y^{j,i}_{k,1})(y+1) + y^{j,i}_{k,1},\ \frac{z - z^{i,j}_{k,1}}{z^{i,j}_{k,2} - z^{i,j}_{k,1}} + 2i - 1 \right),$$

$$L^{i,j}_{k,-}(x,y,z) := \left(S(x),\ \frac{1}{2}(y^{j,i}_{k,2} - y^{j,i}_{k,1})(y+1) + y^{j,i}_{k,1},\ \frac{z - z^{i,j}_{k,2}}{z^{i,j}_{k,1} - z^{i,j}_{k,2}} + 2i - 1 \right),$$

$T^{j,\mathrm{bot}}_{+}(x,y,z) :=$

$$\left(S(x),\ \frac{1}{2}(y_{1,2}^{j,i^j_{\rm bot}} - y_{1,1}^{j,i^j_{\rm bot}})(y+1) + y_{1,1}^{j,i^j_{\rm bot}},\ \frac{z-2j+1}{2(z_{1,1}^{i^j_{\rm bot},j} - 2j+1)} + 2i^j_{\rm bot} - \frac{3}{2} \right),$$

$$T_-^{j,\rm bot}(x,y,z) :=$$

$$\left(S(x),\ \frac{1}{2}(y_{1,2}^{j,i^j_{\rm bot}} - y_{1,1}^{j,i^j_{\rm bot}})(y+1) + y_{1,1}^{j,i^j_{\rm bot}},\ \frac{z - z_{1,1}^{i^j_{\rm bot},j}}{2(2j-1-z_{1,1}^{i^j_{\rm bot},j})} + 2i^j_{\rm bot} \right),$$

$$T_+^{j,\rm top}(x,y,z) :=$$

$$\left(S(x),\ \frac{1}{2}(y_{|a_{i^j_{\rm top},j}|,2}^{j,i^j_{\rm top}} - y_{|a_{i^j_{\rm top},j}|,1}^{j,i^j_{\rm top}})(y+1) + y_{|a_{i^j_{\rm top},j}|,1}^{j,i^j_{\rm top}},\ \frac{z - z_{|a_{i^j_{\rm top},j}|,2}^{i^j_{\rm top},j}}{2(2j - z_{|a_{i^j_{\rm top},j}|,2}^{i^j_{\rm top},j})} + 2i^j_{\rm top} \right),$$

$$T_-^{j,\rm top}(x,y,z) :=$$

$$\left(S(x),\ \frac{1}{2}(y_{|a_{i^j_{\rm top},j}|,2}^{j,i^j_{\rm top}} - y_{|a_{i^j_{\rm top},j}|,1}^{j,i^j_{\rm top}})(y+1) + y_{|a_{i^j_{\rm top},j}|,1}^{j,i^j_{\rm top}},\ \frac{z-2j}{2(z_{|a_{i^j_{\rm top},j}|,2}^{i^j_{\rm top},j} - 2j)} + 2i^j_{\rm top} - \frac{3}{2} \right).$$

Given $x \in \mathbf{R}$, represent

$$\mathrm{sign}(x) = \begin{cases} + & \text{if} \quad x > 0, \\ - & \text{if} \quad x < 0. \end{cases}$$

We start the construction of f by defining it on the sets $H_k^{i,j}$, $A^j_{\rm bot}$ and $A^j_{\rm top}$. Set

$$\begin{aligned} f|_{H_k^{i,j}} &= L^{i,j}_{k,\mathrm{sign}(a_{i,j})}, \\ f|_{A^j_{\rm bot}} &= T^{j,\rm bot}_{\mathrm{sign}(a_{i^j_{\rm bot},j})}, \\ f|_{A^j_{\rm top}} &= T^{j,\rm top}_{\mathrm{sign}(a_{i^j_{\rm top},j})}. \end{aligned}$$

Observe that in each case f is a C^∞-diffeomorphism onto its image. Furthermore $f(H_k^{i,j}) \subset V_k^{j,i}$ and $f(A^j_{\rm bot} \cup A^j_{\rm top}) \subset \mathrm{cl}(X \setminus \bigcup_{j=1}^n D^j)$.

Turning now to the sets $E_k^{i,j}$ we require that

$$f(E_k^{i,j}) \subset X \setminus \bigcup_{j=1}^n \left[D^j \cup f(A^j_{\rm bot}) \cup f(A^j_{\rm top}) \right],$$

$$f|_{E_k^{i,j}}\left(x,y,z_{k,2}^{i,j}\right) = L_{k,\mathrm{sign}(a_{i,j})}^{i,j}\left(x,y,z_{k,2}^{i,j}\right),$$

$$f|_{E_k^{i,j}}\left(x,y,z_{k+1,1}^{i,j}\right) = L_{k+1,\mathrm{sign}(a_{i,j})}^{i,j}\left(x,y,z_{k+1,1}^{i,j}\right),$$

and for all i, i', j, j'

$$f(E_k^{i,j}) \cap f(E_{k'}^{i',j'}) = \emptyset,$$

where $k \neq k'$. The last requirement is that $f : E_k^{i,j} \to f(E_k^{i,j})$ be a C^∞-diffeomorphism. Finally, for the sets $G^{i,j}$, we choose f such that

$$f(G^{i,j}) \subset X \setminus \bigcup_{j=1}^{n}\left[D^j \cup f(A_{\mathrm{bot}}^j) \cup f(A_{\mathrm{top}}^j) \cup \left(\bigcup_{i\in\mathcal{I}^j}\bigcup_{k=1}^{|a_{i,j}|-1} f(E_k^{i,j})\right)\right],$$

$$f|_{G^{i,j}}\left(x,y,z_{|a_{i,j}|,2}^{i,j}\right) = L_{|a_{i,j}|,\mathrm{sign}(a_{i,j})}^{i,j}\left(x,y,z_{|a_{i,j}|,2}^{i,j}\right),$$

$$f|_{G^{i,j}}\left(x,y,z_{1,1}^{l_i^j,j}\right) = L_{1,\mathrm{sign}(a_{l_i^j,j})}^{l_i^j,j}\left(x,y,z_{1,1}^{l_i^j,j}\right),$$

and for all i, i', j, j'

$$f(G^{i,j}) \cap f(G^{i',j'}) = \emptyset.$$

As before, f is extended in such way that it is also a C^∞-diffeomorphism from $G^{i,j}$ onto its image.

The final step is to define the map f on the following slabs:

$$K^j := \{(x,y,z) \in \mathbf{R}^3 \mid 0 \leq x \leq 1,\ -1 \leq y \leq 1,\ 2j \leq z \leq 2j+1\},$$

for $j = 0, \ldots, n$. The map f must satisfies:

$$f(K^j) \subset X \setminus \bigcup_{l=1}^{n}\left[D^l \cup f(A_{\mathrm{bot}}^l) \cup f(A_{\mathrm{top}}^l) \cup \left[\bigcup_{i\in\mathcal{I}^l} f(G^{i,l}) \cup \left(\bigcup_{k=1}^{|a_{i,l}|-1} f(E_k^{i,l})\right)\right]\right],$$

$$f|_{K^j}(x,y,2j) = f|_{A_{\mathrm{top}}^{j-1}}(x,y,2j), \quad \text{for } j = 1, \ldots, n,$$

$$f|_{K^j}(x,y,2j+1) = f|_{A_{\mathrm{bot}}^{j+1}}(x,y,2j+1), \quad \text{for } j = 0, \ldots, n-1,$$

and, for all $j \neq j'$ it is required

$$f(K^j) \cap f(K^{j'}) = \emptyset.$$

Again, f is chosen to be a C^∞-diffeomorphism from K^j onto its image.

It is left to the reader to check that $f : X \to f(X) \subset \operatorname{int}(N)$ is a C^∞-diffeomorphism.

To finish up the first part of the proof, we need to determine the Conley index of an isolated invariant set of the map f. The compact set $N = \bigcup_{j=1}^{n} D^j$ is an isolating neighborhood. Let $S = \operatorname{Inv}(N, f)$. An index pair for S is $(\mathcal{D}, L)$, where

$$
\begin{aligned}
\mathcal{D} &:= \bigcup_{j=1}^{n} \left[D^j \cup f(A^j_{\text{bot}}) \cup f(A^j_{\text{top}}) \cup \left(\bigcup_{i \in \mathcal{I}^j} f(G^{i,j}) \cup \left(\bigcup_{k=1}^{|a_{i,j}|-1} f(E^{i,j}_k) \right) \right) \right], \\
L &:= \bigcup_{j=1}^{n} \left[A^j_{\text{bot}} \cup f(A^j_{\text{bot}}) \cup A^j_{\text{top}} \cup f(A^j_{\text{top}}) \right] \cup \\
&\quad \bigcup_{j=1}^{n} \left[\left(\bigcup_{i \in \mathcal{I}^j} f(G^{i,j}) \cup \left(\bigcup_{k=1}^{|a_{i,j}|-1} f(E^{i,j}_k) \right) \right) \right].
\end{aligned}
$$

This implies that the homotopy type of $(\mathcal{D}/L, [L])$ is $\vee_{p=1}^{n} S^1$, where $\vee$ denotes the wedge product. Therefore,

$$
H^n(\mathcal{D}, L) = \begin{cases} \mathbf{Q}^n & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Notice that $M = F^1_{\mathcal{D},L}$ which implies the spectrum of the index automorphism $\chi^1(S)$ is $\sigma^\times(M)$.

The final step is to construct examples for all positive integers q. The argument is as follows. Consider the following product space $X \times \mathbf{R}^{q-1} \subset \mathbf{R}^{q+2}$ and the map $F : X \times \mathbf{R}^{q-1} \to X \times \mathbf{R}^{q-1}$ defined by

$$
F(x, w) = (f(x), 2w).
$$

Obviously, since f is a C^∞-diffeomorphism F is a C^∞-diffeomorphism. Observe that $N \times [-1,1]^{q-1} \subset X \times \mathbf{R}^{q-1}$ is an isolating neighborhood for F and $S \times \{0\} = \operatorname{Inv}(N \times [-1,1]^{q-1}, F)$. Given

$$
\begin{aligned}
\mathcal{N} &:= \mathcal{D} \times [-1,1]^{q-1}, \\
\mathcal{L} &:= (L \times [-1,1]^{q-1}) \cup (\mathcal{D} \times (\{-1\} \times \mathbf{R}^{q-2})) \cup (\mathcal{D} \times (\{1\} \times \mathbf{R}^{q-2})),
\end{aligned}
$$

we have that $(\mathcal{N}, \mathcal{L})$ is an index pair for $S \times \{0\}$ and for all n,

$$
H^n(\mathcal{N}, \mathcal{L}) \simeq H^{n-q+1}(\mathcal{D}, L).
$$

The final step is to characterize the induced map $F^q_{\mathcal{N},\mathcal{L}}$. Since F is the product of the map f and an expansion map, the index information one obtains is exactly that from the map f. The additional dimensions added do not provide any new invariant set to the map F. In fact, as in the case $q = 1$, the spectrum of the index automorphism $\chi^q(S \times \{0\})$ is $\sigma^\times(M)$. ■

5 Proofs of Proposition 1.5, Theorems 1.3, 1.7 and 1.8

In the examples we need to construct, we will make use of the following matrix associated to a polynomial. Given a polynomial $p(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$ the $m \times m$ matrix

$$C_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{m-3} & -a_{m-1} & -a_{m-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

is called the *companion matrix* associated with the polynomial p. Its characteristic polynomial can be computed:

Proposition 5.1 *If C_p is the companion matrix associated with the polynomial $p(x)$, then*

$$\det(C_p - x\mathrm{Id}_m) = p(x),$$

where Id_m denotes the $m \times m$ identity matrix.

Proof of Theorem 1.3. Let p be an integral polynomial for which the set of non-zero roots is P and apply Proposition 4.1 to the matrix C_p. ■

Proof of Proposition 1.5. Let p_i, $i = 0, 1$ and ϱ be an integral polynomials for which the set of non-zero roots is P_i and $P \setminus (P_0 \amalg P_1)$, respectively. Suppose C_ϱ is a $l \times l$ matrix and C_{p_i} is a $k_i \times k_i$ matrix. Let $t = k_0 + k_1 + l$. Consider the following $t \times t$ matrix:

$$M = \begin{bmatrix} C_{p_0} & 0 & 0 \\ 0 & C_\varrho & 0 \\ 0 & 0 & C_{p_1} \end{bmatrix}$$

and the two projections $\mathcal{P}_i$, $i = 0, 1$, given by

$$\mathcal{P}_0 = \begin{bmatrix} \mathrm{Id}_{k_0+t-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{k_1+1} \end{bmatrix}.$$

By Theorem 1.3, given $q > 0$, there exist a set $X \subset \mathbf{R}^{q+2}$, a C^∞-diffeomorphism $f : X \to f(X) \subset X$ and an isolating neighborhood $N \subset X$ such that $S = \mathrm{Inv}(N, f)$ is hyperbolic and

$$\sigma(\chi^n(S)) = \begin{cases} P & \text{if } n = q, \\ \emptyset & \text{otherwise.} \end{cases}$$

The next step is to prove that we can find an isolating neighborhood $N = N_0 \cup N_1$, where N_0 and N_1 are disjoint compact sets. In the proof of Theorem 4.1, a construction of an isolating neighborhood was given. Since

$$\sum_{\lambda \in P \setminus (P_0 \amalg P_1)} \lambda = 0,$$

the $(t-1, t-1)$ entry of the matrix C_ϱ is zero. Therefore, the compact set $\mathcal{D}$ constructed in the proof of Theorem 4.1 is the union of two disjoint compact sets which will be called $\mathcal{D}_0$ and $\mathcal{D}_1$. These compact sets induce a decomposition for the isolating neighborhood N denoted by $\{N_0, N_1\}$. Moreover, if $S_i = \mathrm{Inv}(N_i, f)$, the construction described in the proof of Theorem 4.1 shows that there exist compact sets L_0 and L_1 such that $(\mathcal{D}_i, L_i)$ is an index pair for S_i and

$$\sigma(\chi^n(S_i)) = \begin{cases} \sigma^\times(\mathcal{P}_i M) = P_i & \text{if } n = q, \\ \emptyset & \text{otherwise,} \end{cases}$$

which concludes the proof. ■

Proof of Theorem 1.7. As in the proof of Proposition 1.5, the first step is to construct a matrix that encodes the desired properties. Therefore, let p_i, $i = 0, 1$ be an integral polynomial for which the set of non-zero roots is P_i. Suppose C_{p_i} is a $k_i \times k_i$ matrix and let $k = k_0 + k_1$. The $k \times k$ matrix M is defined by

$$M = \begin{bmatrix} C_{p_0} & 0 \\ 0 & C_{p_1} \end{bmatrix}.$$

We also consider the two projections $\mathcal{P}_i$, $i = 0, 1$,

$$\mathcal{P}_0 = \begin{bmatrix} \mathrm{Id}_{k_0} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{k_1} \end{bmatrix}.$$

Now, it follows as the previous proof. ■

Proof of Theorem 1.8. Since $P\setminus(P_0 \amalg P_1)$ is a cyclic set, there exist $s \in \mathbf{Z}^+$, $r_i \in \mathbf{Z}^+$, $r_i > 1$ for $i = 1, \ldots, s$, and $\nu_i \in \mathbf{Z}$ such that

$$P \setminus (P_0 \amalg P_1) := \amalg_{i=1}^{s}\{\nu_{ij} \mid \nu_{ij}^{r_i} = \nu_i, j = 1, \ldots, r_i\}.$$

Consider the $r_i \times r_i$ matrix C_{ϱ_i}, where $\varrho_i(x) = x^{r_i} - \nu_i$, $i = 1, \ldots, s$. As before, p_i, $i = 0, 1$, denotes the integral polynomial for which the set of non-zero roots is P_i. We assume C_{p_i} is a $k_i \times k_i$ matrix. Finally, let $t = k_0 + l + k_1$, where $l = \sum_{i=1}^{s} r_i$. Three $t \times t$ matrices are defined as follows. Set

$$M = \begin{bmatrix} C_{p_0} & 0 & \cdots & 0 & 0 \\ 0 & C_{\varrho_1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & C_{\varrho_s} & 0 \\ 0 & 0 & \cdots & 0 & C_{p_1} \end{bmatrix},$$

$$\mathcal{P}_0 = \begin{bmatrix} \mathrm{Id}_{k_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Id}_{r_1} - \mathrm{J}_{r_1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{r_2} - \mathrm{J}_{r_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{Id}_{r_s} - \mathrm{J}_{r_s} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$\mathcal{P}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{J}_{r_1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{J}_{r_s} & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{Id}_{k_1} \end{bmatrix},$$

where J_{r_i} is the $r_i \times r_i$ matrix such that the entry (r_i, r_i) is equal to 1 and zero everywhere else. The conclusions about the index properties follow as in Proposition 1.5.

We now turn to the analysis of $S \setminus (S_0 \cup S_1)$. Notice that for each, $i = 0, \ldots, s$, we can determine a projection $\mathcal{R}_i$ with the property that there exists an isolating neighborhood $\mathcal{N}_i = \bigcup_{j=1}^{r_i} \mathcal{K}_{ij}$ such that for each $i = 1, \ldots, s$, $\mathcal{K}_{ij}$ are comp act and disjoint sets and

$$\sigma(\chi^n(\mathcal{S}_i)) = \begin{cases} C_{\varrho_i} & \text{if } n = q, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\mathcal{S}_i = \operatorname{Inv}(\mathcal{N}_i, f)$. Moreover,

$$S \setminus (S_0 \cup S_1) = \bigcup_{i=1}^{s} \mathcal{S}_i.$$

Notice that $(C_{\varrho_i})^{r_i} = \mu_i \mathrm{Id}$. If $x \in S_i = \operatorname{Inv}(\mathcal{N}_i, f)$, then $x \in \mathcal{K}_{ij_0}$ for some j_0. Moreover, $(f^{r_i})^n(x) \in \mathcal{K}_{ij_0}$ for all non-negative n. Since $\mathcal{K}_{ij_0}$ is compact,

$$f^{r_i n}(x) \to \overline{x},$$

where $\overline{x} \in \mathcal{K}_{ij_0}$. Since $f^{r_i(n+1)}(x)$ also converges to $\overline{x}$ and f^{r_i} is continuous, $f^{r_i}(\overline{x}) = \overline{x}$. Moreover, there does not exist a non-negative integer k, $k < r_i$, such that $f^k(\overline{x}) = \overline{x}$. In fact, if $k < r_i$, then $f^k(\overline{x}) \notin \mathcal{K}_{ij_0}$, which is a contradiction. ■

6 Proof of Proposition 1.10

We start with the following example. Let

$$X := \left(\bigcup_{i=1}^{3} Q_i\right) \cup \left(\bigcup_{i=1}^{2} R_i\right) \cup \left(\bigcup_{i=1}^{3} S_i\right)$$

be a union of squares in $\mathbf{R}^2$ as pictured in Figure 1. Consider a C^∞-diffeomorphism $f : X \to f(X) \subset X$ defined as follows. On $\cup_{i=1}^3 Q_i$ and S_2 the definition of f is as in Figure 1. We require that $f(R_3) \subset S_2$. Moreover, it is also required that $f(R_1) \subset S_1$, $f(R_2) \subset S_3$, $f(S_1) \subset R_1$, and $f(S_2) \subset R_2$, $f(S_3) \subset R_1$.

An isolating neighborhood for f is $N = \cup_{i=1}^3 Q_i$. If we consider (N, L), where

$$L := \left(\bigcup_{i=1}^{2} R_i\right) \cup \left(\bigcup_{i=1}^{3} S_i\right) \cup R_3,$$

we have satisfied the conditions for index pair for $S = \operatorname{Inv}(N, f)$ and

$$F^1_{N,L} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{7}$$

If $N_0 := Q_1$ and $N_1 := Q_2 \cup Q_3$, then $\{N_0, N_1\}$ is a decomposition for N. Moreover,

$$\sigma(\chi^n(S)) = \sigma(\chi^n(S_0)) = \sigma(\chi^n(S_1)) = \emptyset.$$

However a horseshoe is still present. More precisely,

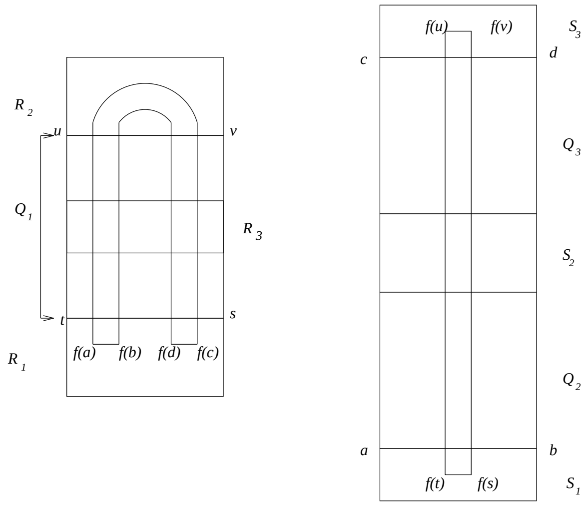

Figure 1: The definition of the map f on X.

Proposition 6.1 *Let f and N be as defined above. There exists a continuous surjective map $\rho : S \to \Sigma_2^+$ such that*

$$T \circ \rho = \rho \circ f^2.$$

PROOF. We need to define the map ρ. First notice that

$$(F^1_{N,L})^2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Also, if $x \in S$ and $x \in N_i$ then $f^{2n}(x) \in N_i$ for all $n \in \mathbf{Z}^+$. These facts allow us to determine $\{N_i^0, N_i^1\}$ decomposition of N_i, $i = 0, 1$. Thus, define the functions $\theta, \Theta_i : S \to \{0, 1\}$ by

$$\theta(x) = \begin{cases} 0 & \text{if } x \in N_0, \\ 1 & \text{if } x \in N_1, \end{cases} \quad \text{and} \quad \Theta_i(x) = \begin{cases} 0 & \text{if } x \in N_i^0, \\ 1 & \text{if } x \in N_i^1. \end{cases}$$

Finally, set $\rho : S \to \Sigma_2^+$ by

$$\rho(x) := (\Theta_{\theta(x)}(x), \Theta_{\theta(x)}(f^2(x)), \Theta_{\theta(x)}(f^4(x)), \Theta_{\theta(x)}(f^6(x)), \ldots).$$

The function ρ is well defined and the proof that ρ is a semi-conjugacy is as in [2, Theorem 1.1]. ■

Proof of Proposition 1.10. As in Proposition 1.5, let p_i, $i = 0,1$ and ϱ be integral polynomials for which the multiset of nonzero roots is P_i and $P \setminus (P_0 \amalg P_1)$, respectively. Suppose C_ϱ is an $l \times l$ matrix and C_{p_i} is a $k_i \times k_i$ matrix and K is the matrix defined in (7). Let $t = k_0 + k_1 + l + 4$. First, define the following three $t \times t$ matrices:

$$M = \begin{bmatrix} C_{p_0} & 0 & 0 & 0 \\ 0 & C_\varrho & 0 & 0 \\ 0 & 0 & C_{p_1} & 0 \\ 0 & 0 & 0 & K \end{bmatrix},$$

$$\mathcal{P}_0 = \begin{bmatrix} \mathrm{Id}_{k_0} & 0 & 0 & 0 \\ 0 & \mathrm{Id}_l - \mathrm{J}_l & 0 & 0 \\ 0 & 0 & 0_{k_1} & 0 \\ 0 & 0 & 0 & 0_4 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_1 = \begin{bmatrix} 0_{k_0} & 0 & 0 & 0 \\ 0 & \mathrm{J}_l & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{k_1} & 0 \\ 0 & 0 & 0 & \mathrm{Id}_4 \end{bmatrix}.$$

Hence, $\sigma^\times(M) = P$. By Theorem 1.3, given $q > 0$, there exist a set $X \subset \mathbf{R}^{q+2}$, a C^∞-diffeomorphism $f : X \to f(X) \subset X$, and an isolating neighborhood $N \subset X$ such that $S = \mathrm{Inv}(N, f)$ is hyperbolic and

$$\sigma(\chi^n(S)) = \begin{cases} P & \text{if } n = q, \\ \emptyset & \text{otherwise.} \end{cases}$$

As in the proof of Proposition 1.5, there exists a decomposition for N, $\{N_0, N_1\}$, such that

$$\sigma(\chi^q(S_i)) = \sigma^\times(\mathcal{P}_i M) = P_i.$$

The definition of the matrix M implies that the set $\mathcal{D}$ constructed in the proof of Proposition 4.1 has three connected components. Moreover, the set $\mathcal{D}_0$ (defined in the proof of Proposition 1.5) has one connected component and $\mathcal{D}_1$ (also defined in the proof of Proposition 1.5) has two connected components, where one of them can be associated with the matrix K. Call it $\mathcal{D}_{11}$. Notice that $f(\mathcal{D}_{11}) \subset \mathcal{D}_{11}$. One more fact can be inferred from the construction described in Propositions 4.1 and 1.5: the isolating neighborhood N_1 is a disjoint union of two compact sets. Call them N_{10} and N_{11}, with N_{11} being an isolating neighborhood for f restricted to $\mathcal{D}_{11}$. The isolating neighborhood N_{11} induces the following projection:

$$\mathcal{P}_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_4 \end{bmatrix},$$

and we use it to compute $\sigma(\chi^q(S_{11}))$, where $S_{11} = \text{Inv}(N_{11}, f)$. The following is valid:

$$\sigma(\chi^q(S_{11})) = \sigma^\times(\mathcal{P}_{11}M).$$

By Proposition 6.1, S_{11} is semiconjugate to Σ_2^+. Therefore, to finish the proof of the result, define $S' := S_{11}$. ■

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