Dr. Z's Math152 Handout #11.12 [Applications of Taylor Polynomials]

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Problem Type 11.12a: (a) Approximate f by a Taylor polynomial of degree n at the number a.

(b) Use Taylor's inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

Example Problem 11.12a: Ditto with

$$f(x) = x^{2/3} \ , \ a = 1 \ , \ n = 3 \ , \ 0.8 \leq x \leq 1.2$$

Steps	**Example**
1. Write down the general Taylor polynomial of degree n, and then implement it for the function and the center a $T_n(x) = f(a)+f'(a)(x-a)+f''(a)(x-a)^2/2+$ $f'''(a)(x-a)^3/6+\ldots+f^{(n)}(a)(x-a)^n/n! \quad .$	**1.** $T_n(x) = f(a)+f'(a)(x-a)+f''(a)(x-a)^2/2$ $+f'''(a)(x-a)^3/6+\ldots+f^{(n)}(a)(x-a)^n/n! \quad .$ Plugging-in $n = 3$ and $a = 1$, we get $T_3(x) = f(1)+f'(1)(x-1)+f''(1)(x-1)^2/2$ $+f'''(1)(x-1)^3/6 \quad .$ Since in this problem, $f(x) = x^{2/3}$, $f'(x) = (2/3)x^{-1/3}$, $f''(x) = (2/3)(-1/3)x^{-4/3} = (-2/9)x^{-4/3}$, $f'''(x) = (8/27)x^{-7/3}$. Plugging-in $x = 1$, $f(1) = 1$, $f'(1) = 2/3$, $f''(1) = -2/9$, $f'''(1) = 8/27$, and we get **Ans. to part (a)**: $T_3(x) = 1+\frac{2}{3}(x-1)-\frac{1}{9}(x-1)^2+\frac{4}{81}(x-1)^3 \quad .$

2. Write down (or look up from the formula sheet), Taylor's inequailty for the remaineder.

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad ,$$

where M is the maximum of $f^{n+1}(x)$ in the desiganted interval. Plug-in the specfic a and n.

2.

$$|R_3(x)| \leq \frac{M}{4!}|x-1|^4 \quad ,$$

Since (from above), $f'''(x) = (8/27)x^{-7/3}$, we have $f^{(4)}(x) = (-56/81)x^{-10/3}$, so $|f^{(4)}(x)| = (56/81)|x|^{-10/3}$. This is decreasing, so its maximum is on the left when $x = .8$, so $M = (56/81)/(.8)^{10/3} = 1.455$. Also the interval can be written $|x-1| \leq .2$ so $|x-1|^4 \leq (.2)^4$. Combining everything,

$$|R_3(x)| \leq \frac{1.455}{24}(.2)^4 = 0.000097$$

Ans. to (b): The error in using $T_3(x)$ instead of $f(x) = x^{2/3}$ in the interval $0.8 \leq x \leq 1.2$ is **always** less than 0.000097.