

Theorem: $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$

This theorem says that $\lim_{x \rightarrow a} f(x)$ exists if and only if:

1. $\lim_{x \rightarrow a^+} f(x)$ exists and
2. $\lim_{x \rightarrow a^-} f(x)$ exists and
3. Both limits in 1. and 2. are equal.

Definition: f is **continuous** at a if the following conditions are satisfied :

1. $f(a)$ is defined.
2. $\lim_{x \rightarrow a} f(x)$ exists
3. $\lim_{x \rightarrow a} f(x) = f(a)$

Definition: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ so $f'(a) = f'(x)|_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

f is **differentiable** at a if $f'(a)$ exists.

Definition: The **differential dy** is: $dy = f'(x)dx$. Compare with $\Delta y = f(x + \Delta x) - f(x)$.
 $f(a + \Delta x) \approx f(a) + dy = f(a) + f'(a)dx$

Graphing and Optimization. Local max and min of a function f can occur only at **critical points** (any point x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist). **Absolute max and min** occur only at critical points or endpoints. **Inflection points** are points $(x, f(x))$ where the concavity of f changes and can occur only where $f''(x)$ is zero or does not exist. The line $x = a$ is a **vertical asymptote** of the graph of a function f if $\lim_{x \rightarrow a^+} = +\infty$ or $-\infty$, or if

$$\lim_{x \rightarrow a^-} = +\infty \text{ or } -\infty,$$

The line $y = b$ is a **horizontal asymptote** of the graph of f if $\lim_{x \rightarrow +\infty} f(x) = b$ or $\lim_{x \rightarrow -\infty} f(x) = b$

Basic Fact: $f'(a)$ is the **slope of the tangent line** to the graph of f at $x = a$

Basic log and exp laws: $e^{\ln x} = x$ for $x > 0$ $\ln e^x = x$ for all real numbers x
 $\ln(mn) = \ln m + \ln n$ $\ln \frac{m}{n} = \ln m - \ln n$ $\ln m^n = n \ln m$ $\ln 1 = 0$ $\ln e = 1$
 $e^a e^b = e^{a+b}$ $\frac{e^a}{e^b} = e^{a-b}$ $(e^a)^b = e^{ba}$ $e^0 = 1$ $e^1 = e$

Basic Trig Identities: $\sin^2 x + \cos^2 x = 1$ $\sin(-x) = -\sin x$ $\cos(-x) = \cos x$
 $\sin(2\pi + x) = \sin x$ $\cos(2\pi + x) = \cos x$ 360 degrees = 2π radians.
 $\tan x = \frac{\sin x}{\cos x}$ $\cot x = \frac{\cos x}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\csc x = \frac{1}{\sin x}$

Definition: $f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$ is a **Riemann sum** for a function f corresponding to a partition of the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{(b-a)}{n}$, if x_1 is in the first subinterval, x_2 is in the second subinterval, etc.

Definition: The **definite integral of f from a to b** , written $\int_a^b f(x) dx$, is defined to be the limit as $n \rightarrow \infty$ of such Riemann sums, if the limit exists (for all choices of representative points x_1, x_2, \dots, x_n in the n subintervals).

$$\text{Thus, } \int_a^b f(x) dx = \lim_{n \rightarrow \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x]$$

Fact: If f is continuous and nonnegative on $[a, b]$, then the **area of the region under the graph of f on $[a, b]$** is the definite integral $\int_a^b f(x) dx$. If f is sometimes negative on $[a, b]$ then $\int_a^b f(x) dx$ is equal to the **area of the region above $[a, b]$ minus the area of the region below $[a, b]$** .

The Fundamental Theorem of Calculus: Let f be continuous on the closed interval $[a, b]$, then $\int_a^b f(x) dx = F(b) - F(a)$ where F is any antiderivative of f (that is $F'(x) = f(x)$).

Theorem: Let G be an antiderivative of f on an interval I . Then every antiderivative F of f on I must be of the form $F(x) = G(x) + C$ where C is a constant.

Definition: The **average value** of an integrable function f over $[a, b]$ is: $\frac{1}{b-a} \int_a^b f(x) dx$

Differentiation Rules

and

Integration Rules

$$(kx)' = k$$

$$\int k dx = kx + C$$

$$(cf(x))' = cf'(x)$$

$$\int cf(x) dx = c \int f(x) dx$$

$$(x^r)' = rx^{r-1}$$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C \text{ for } r \neq -1$$

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

$$(e^x)' = e^x$$

$$\int e^x dx = e^x + C$$

$$(\ln x)' = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$(\sin x)' = \cos x$$

$$\int \cos x dx = \sin x + C$$

$$(\cos x)' = -\sin x$$

$$\int \sin x dx = -\cos x + C$$

$$(\tan x)' = \sec^2 x$$

$$\int \sec^2 x dx = \tan x + C$$

Chain Rule: If $h(x) = g[f(x)]$, then $h'(x) = g'(f(x)) * f'(x)$. Equivalently, $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$, where we have written $u = f(x)$ and $y = h(x) = g(u)$.

Integration by Substitution: Let $u = g(x)$ and $F(u)$ be the antiderivative of $f(u)$. Then $du = g'(x)dx$ and $\int f(g(x))g'(x) dx = \int f(u) du = F(u) + C$

$$\text{Also, } \int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u) du = F(g(b)) - F(g(a))$$

Product Rule: $[f(x) * g(x)]' = f'(x) * g(x) + g'(x) * f(x)$

Quotient Rule: $\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x) * f'(x) - f(x) * g'(x)}{g^2(x)}$

Properties of the Definite Integral

$$\int_a^a f(x) dx = 0 \text{ (same integration limits)} \quad \int_a^b f(x) dx = -\int_b^a f(x) dx \text{ (exchange integration limits)}$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \text{ where } a < c < b.$$

Calculus in Economics Definitions, from section 3.4: Profit P equals revenue R minus total cost C . Revenue $R = px$. Average cost is $\bar{C}(x) = \frac{C(x)}{x}$. The demand equation relates price per unit p and number of units x . It can be solved for p as a function of x , or x as a function of p .

The elasticity of demand of x as a function of p , $x = f(p)$, is given by $E(p) = -\frac{pf'(p)}{f(p)}$. The demand is **elastic** (price increase causes revenue to decrease) if $E(p) > 1$, **inelastic** (price increase causes revenue to increase) if $E(p) < 1$, and **unitary** if $E(p) = 1$.