Answers to Review Problems for Calculus 135 Exam I

Short answers are listed first. Detailed answers are given following the list of short answers.

Short Answers

1 a. $g(f(x)) = \frac{1}{\sqrt{x^2+3x-40}}$
The domain is: $(-\infty,-8)\bigcup(5,+\infty)$
The derivative is :$[g(f(x))]' - \frac{1}{2}(x^2+3x-40)^{-\frac{3}{2}}(2x+3)$

1. b $[f(g(x))]' = \frac{1}{2}(x^{-2}+3x^{-1}-40)^{-\frac{1}{2}}(-2x^{-3}-3x^{-2})$

2 a. $-\frac{1}{4}$

2 b. $\frac{1}{6}$

2 c. $\frac{3}{2}$

3. See detailed answers.

4. $g(3) = 1.5 \qquad g'(3) = -.5$

5 a. $C = 3$

5 b. Yes.

5 c. 10, $\quad -1$

5 d. No.

5 e. See detailed answers.

6. $2x$

7. See detailed answers.

8. $y = -5x + 11$

9. 60

10 a. $f'(x) = \frac{3}{5}x^{-\frac{2}{5}} - \frac{18}{8}x^{-\frac{11}{8}} + 3x^2$

10 b. $g'(x) = (x+3x^9)(5x^4-7) + (1+27x^8)(x^5-7x)$

10 c. $h'(x) = 5\left(\frac{x^2+3x}{x^6-9x}\right)^4\left(\frac{(x^6-9x)(2x+3)-(x^2+3x)(6x^5-9)}{(x^6-9x)^2}\right)$

10 d. $k'(x) = \frac{1}{3}(x^7+5x^2-50)^{-\frac{2}{3}}(7x^6+10x)$

11 a. $F'(x) = \frac{5}{2}x^{-\frac{1}{2}} \qquad F''(x) = -\frac{5}{4}x^{-\frac{3}{2}}$

11 b. 0

12 a. $R(x) = 120x - 0.001x^2$

12 b. $P(x) = -0.001x^2 + 116x - 6000$

12 c. $P'(x) = -0.002x + 116$

12 d. \$108.

12 e. The demand is inelastic when $p = 30$ and elastic when $p = 90$.

13. The radius is increasing at the rate of $\frac{1}{10\pi}$ cm/sec. The surface area is increasing at the rate of 4 cm^2/sec.

14. See detailed answers.

Detailed Answers

1 a. The composition $g(f(x)) = g(\sqrt{x^2+3x-40}) = \frac{1}{\sqrt{x^2+3x-40}}$ or $(x^2+3x-40)^{-\frac{1}{2}}$. The domain of $g(f(x))$ is the set of those values of x for which $x^2+3x-40$ is not zero, and is not negative. To find these values of x we first determine the values of x at which $x^2+3x-40$ is equal to zero. Now, $x^2+3x-40 = (x+8)(x-5)$, so the values of x at which $x^2+3x+40$ is equal to zero are $x=-8$ and $x=5$. Next we test the sign of $x^2+3x-40$ at sample points in the intervals determined by the points $x=-8$ and $x=5$. Testing sample points in each of the intervals: $(-\infty-8), (-8,5)$ and $(5,\infty)$, we find that:

$x^2+3x-40$ is positive on $-(\infty,-8)$ and $(5,\infty)$ and negative on $(-8,5)$.
The domain of $g(f(x))$ then would be: $(-\infty,-8)\bigcup(5,+\infty)$

To find the derivative of $g(f(x))$ we use the chain rule:

$$[g(f(x))]' = g'(f(x))f'(x) = -\frac{1}{2}(x^2+3x-40)^{-\frac{3}{2}}(2x+3)$$

1 b. The composition $f(g(x)) = f(\frac{1}{x}) = \sqrt{(\frac{1}{x})^2+\frac{3}{x}-40} = (x^{-2}+3x^{-1}-40)^{\frac{1}{2}}$
To find the derivative of $f(g(x))$ we use the chain rule:

$$[f(g(x))]' = f'(g(x))f'(x) = \frac{1}{2}(x^{-2}+3x^{-1}-40)^{-\frac{1}{2}}(-2x^{-3}-3x^{-2})$$

2. Analytic Solutions:

2 a. If we tried substitution, both the numerator and denominator would equal zero, so we must do some algebra first. Factoring both the numerator and denominator we get:

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4} = \lim_{x\to 2}\frac{(x-3)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2}\frac{(x-3)}{(x+2)} = -\frac{1}{4}$$

2b. Using substitution we get:

$$\lim_{x\to 4}\frac{x^2-5x+6}{x^2-4} = \frac{4^2-5(4)+6}{4^2-4} = \frac{2}{12} = \frac{1}{6}$$

2c. The numerator and denominator are polynomials of the same degree. The coefficient of the x^4 term in the numerator is 3. The coefficient of the x^4 term in the denominator is 2. If we divide both the numerator and denominator by the highest power of x, namely x^4 we would be left with:

$$\lim_{x\to\infty}\frac{3+\frac{10}{x^3}+\frac{7}{x^4}}{2-\frac{5}{x}} = \frac{3}{2}$$

3. In this problem we use the Intermediate Value Theorem. In order to apply a theorem to a function, we must first check to see if the function meets the conditions in the hypothesis of the theorem. The hypothesis is the “if” part of the theorem. Once we know a function satisfies the hypothesis of a theorem, we can apply the conclusion of the theorem to the function. The conclusion is the "then" part of the theorem. For the Intermediate Value Theorem, the hypothesis is the part that reads "If f is a continuous function on a closed interval $[a, b]$ and M is any number between $f(a)$ and $f(b)$". The conclusion of the Intermediate Value Theorem is the part that reads "then there is at least one number c in $[a, b]$ such that $f(c) = M$".

Now, the function $g(x)$ is a polynomial so it is continuous on the entire real line. Thus it is continuous on the closed interval $[-5, 1]$. The closed interval $[-5, 1]$ serves as the closed interval "$[a, b]$" in the hypothesis of the theorem. This means that $g(-5)$ serves as the "$f(a)$" and $g(1)$ serves as the "$f(b)$" of the theorem. Now, $g(-5) = -30$ which is negative, and $g(1) = 102$ which is positive. Further, the number 0 (which serves as our "M" of the hypothesis) is between $g(-5)$ and $g(1)$. This means that $g(x)$ satisfies the hypothesis of the Intermediate Value Theorem over the interval $[-5, 1]$, and so we can apply the conclusion of the Intermediate Value Theorem. That is, there must be at least one number c in $[-5, 1]$ such that $f(c) = 0$.

4. We can begin by writing the tangent line $2y + x = 6$ at $x = 3$ in the more familiar slope intercept form which is $(y = mx + b)$. This gives us the tangent line as: $y = -.5x + 3$. This allows us to readily read off the slope of the tangent line which is $-.5$.

Now, the point $(3, g(3))$ lies on both the curve $y = g(x)$ and the tangent line. Thus the y value for the point at which $x = 3$ must be the same for both the tangent line and the curve. So at $x = 3$, we have $y = g(3) = -.5(3) + 3 = 1.5$.

The value $g'(3)$ gives the slope of the tangent line to the curve $y = g(x)$ at $x = 3$. Since the tangent line $y = -.5x + 3$ has slope $-.5$, $g'(3)$ must be equal to $-.5$.

5 a. $C = 3$. In order for $f(x)$ to be continuous at $x = 1$, the following conditions must be satisfied:

1. $f(1)$ must exist.
2. $\lim_{x \to 1} f(x)$ must exist.
3. $\lim_{x \to 1} f(x)$ must equal $f(1)$.

We begin with condition number 1. By the definition of $f(x)$ that we are given, $f(1) = C$. So condition number 1 will be satisfied for any finite number C. Next we check condition number 2. To determine if $\lim_{x \to 1} f(x)$ exists, we check the left and right hand limits:
First we check the left hand limit:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 5x - 2 = 3$$

We used the rule "$5x - 2$" because that is the rule that applies for values of x "close to" 1 but less than 1.

Next we check the right hand limit:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^3 + 2 = 3$$

We used the rule "$x^3 + 2$" because that is the rule that applies for values of x "close to" 1 but greater than 1.
Now both the left hand limit = 3 and the right hand limit = 3, so the $\lim_{x\to 1} f(x) = 3$. So condition number 2 is satisfied.
Finally, we need to check condition number 3. We need $\lim_{x\to 1} f(x) = f(1) = C$. If we let $C = 3$ and then condition number 3 will be satisfied and then all of our conditions will be satisfied and our function will be continuous at 1.

5 b. YES. In order for $f(x)$ to be continuous at $x = 0$, we need the following:

1. $f(0)$ must exist.
2. $\lim_{x\to 0} f(x)$ must exist.
3. $\lim_{x\to 0} f(x)$ must equal $f(0)$.

We begin with condition number 1. By the definition of $f(x)$ that we are given $f(0) = 5(0) - 2 = -2$. So $f(0)$ exists, and condition number 1 is satisfied. Next we check condition number 2. To determine if $\lim_{x\to 1} f(x)$ exists, we check the left and right hand limits. First we check the left hand limit:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 - 2 = -2$$

We used the rule "$x^2 - 2$" because this rule applies for all x "close" to 0 but less than 0.
Next we check the right hand limit:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 5x - 2 = -2$$

We used the rule "$5x - 2$" because this rule applies for all x "close" 0 to but greater than 0.
Now both the left hand limit and the right hand limit $= -2$, so the $\lim_{x\to 0} f(x) = -2$ and condition number 2 is satisfied.
Finally we check condition number 3. Now, $\lim_{x\to 0} f(x) = f(0) = -2$, so condition number 3 is satisfied and so all of our conditions are satisfied and $f(x)$ is continuous at 0.

5 c. To find $\lim_{x\to 2} f(x)$ we would ordinarily check the left hand limit and right hand limit individually. Notice however, that the rule $x^3 + 2$ can be used for both the left and right hand limits because this rule applies for all x greater than 1, so it applies for all x "close to" 2, and approaching 2 from the left as well as from the right. So we get:

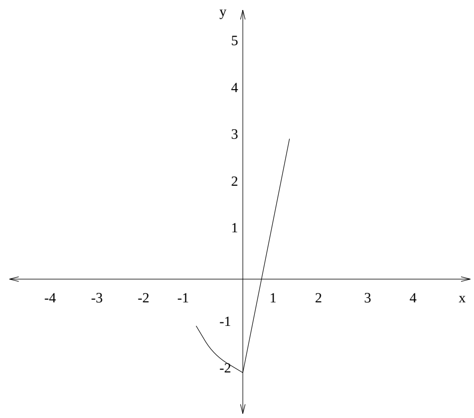
$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x^3 + 2 = 10$$

To find $\lim_{x\to -1} f(x)$ we would ordinarily check both the left hand limit and the right hand limit individually. Notice however, that the rule $x^2 - 2$ can be used for both the left and right hand limits because this rule applies for all x less than zero, so it applies for all x "close to" -1 and approaching -1 from the left as well as from the right. So we get:

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} x^2 - 2 = -1$$

5 d. No, the function is not differentiable at $x = 0$. The value of $f'(x)$ for x on the open interval $(-\infty, 0)$ is given by $2x$. As x "gets close to 0" from the left, the derivative $2x$ be negative and would approach the value 0. The value of $f'(x)$ for x on the open interval $(0, 1)$ on the other hand is the constant 5, so as x gets "close" to 0 from the right, the derivative would have the value 5. Since the derivative has an abrupt change of value from one side of zero to the other, the derivative does not exist at $x = 0$.

5 e. The function is continuous at $x = 0$ because there are no holes, jumps, or breaks in the graph at the point $(0, f(0)) = (0, -2)$. The function is not differentiable at $x = 0$ because there is a sharp corner in the graph at the point $(0, -2)$.



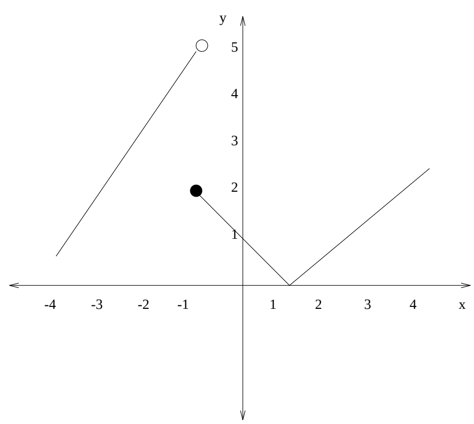
graph of $f(x)$ on the interval $[-1, 1)$

6. $f(x) = x^2 + 2 \qquad f(x + \Delta x) = x^2 + 2x\Delta x + \Delta x^2 + 2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + 2 - (x^2 + 2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

7. There are many possible answers to this question. The following graph is a graph of a function which is continuous at all x in $[-4, 4]$ except at $x = -1$. The function is differentiable at all x in $[-4, 4]$ except at $x = -1$ where the function is discontinuous, and at $x = 1$ where there is a sharp corner in the graph.
The $\lim_{x \to -1^-} f(x) = 5$ and the $\lim_{x \to -1^+} f(x) = 2$



8. To find the equation of the tangent line to the given curve at the point $(2, 1)$ we need to find the slope of the tangent line at the point $(2, 1)$. To find the slope of the tangent line at the point $(2, 1)$, we find the first derivative of the curve, and evaluate the first derivative at $x = 2$ and $y = 1$. We differentiate the function term by term, remembering to use the product rule on the product xy. This gives us:

$$2yy' - [xy' + y] = 2x + y'$$

Next we group all the terms with y' on one side of the equation, and all the terms without y' on the other side of the equation. This gives us:

$$2yy' - xy' - y' = 2x + y$$

Now we factor out the y' giving us:

$$y'[2y - x - 1] = 2x + y$$

Finally, solve for y' as follows:

$$y' = \frac{2x + y}{2y - x - 1}$$

To find the slope of the tangent line at $(2, 1)$, we evaluate y' using $x = 2$ and $y = 1$. The slope of the tangent line at $(2, 1)$ is:

$$\frac{2(2) + 1}{2(1) - 2 - 1} = -\frac{5}{1} = -5$$

Since we have the point $(2, 1)$ and the slope "-5", we can readily give the point-slope form of the tangent line:

$$(y - 1) = -5(x - 2)$$

If we want to give the slope-intercept form of the tangent line, we can write $y = -5x + 11$.

9. In this problem, we use the chain rule. Now, $h'(x) = f'(g(x))g'(x)$.

Thus we get: $h'(1) = f'(g(1))g'(1) = f'(2)(12) = (5)(12) = 60$.

10 a. We rewrite $f(x)$ as $x^{\frac{3}{5}} + 6x^{-\frac{3}{8}} + x^3 = 7$. Next we differentiate term by term using the power rule:

$$f'(x) = \frac{3}{5}x^{-\frac{2}{5}} - \frac{18}{8}x^{-\frac{11}{8}} + 3x^2$$

10 b. We use the product rule to differentiate $g(x)$ giving us:

$$g'(x) = (x + 3x^9)\frac{d}{dx}(x^5 - 7x) + (x^5 - 7x)\frac{d}{dx}(x + 3x^9)$$

$$g'(x) = (x + 3x^9)(5x^4 - 7) + (x^5 - 7x)(1 + 27x^8)$$

10 c. We use the chain rule and the quotient rule to differentiate $h(x)$ giving us:

$$h'(x) = 5\left(\frac{x^2+3x}{x^6-9x}\right)^4 \left(\frac{(x^6-9x)\frac{d}{dx}(x^2+3x) - (x^2+3x)\frac{d}{dx}(x^6-9x)}{(x^6-9x)^2}\right)$$

$$h'(x) = 5\left(\frac{x^2+3x}{x^6-9x}\right)^4 \left(\frac{(x^6-9x)(2x+3) - (x^2+3x)(6x^5-9)}{(x^6-9x)^2}\right)$$

10 d. We rewrite $k(x)$ as $(x^7+5x^2-50)^{\frac{1}{3}}$. Then we use the chain rule to differentiate $k(x)$ giving us:

$$k'(x) = \frac{1}{3}(x^7+5x^2-50)^{-\frac{2}{3}}(7x^6+10x)$$

11 a. We rewrite $F(x) = 5\sqrt{x}$ as $F(x) = 5x^{\frac{1}{2}}$. Then we use the power rule to get:
$F'(x) = \frac{5}{2}x^{-\frac{1}{2}}$ $\qquad$ $F''(x) = -\frac{5}{4}x^{-\frac{3}{2}}$

11 b. Since $g(t)$ is a polynomial of degree 7, the 9^{th} derivative $\frac{d^9g}{dt^9}$ is 0.

12 a. The revenue is given by x times the unit price function $f(x)$ so we get:

$$R(x) = x(120 - 0.001x) = 120x - 0.001x^2$$

12 b. Now $P(x) = R(x) - C(x)$, so we get:

$$P(x) = 120x - 0.001x^2 - (4x + 6000) = -0.001x^2 + 116x - 6000$$

12 c. The Marginal Profit is given by the first derivative of the profit function:

$$P'(x) = -0.002x + 116$$

12 d. The approximate actual profit realized from the sale of the 4001^{st} telescope is: $P'(4000) = -0.002(4000) + 116 = 108$ dollars.

12 e. First, solve the given demand equation for x in terms of p, we find

$$x = f(p) = 120,000 - 1,000p$$

from which we find that $f'(p) = -1000$. Therefore,

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{-1000p}{120,000 - 1,000p} = \frac{p}{120 - p}$$

$E(30) = \dfrac{30}{120-30} = \dfrac{1}{3} < 1$. $E(90) = \dfrac{90}{120-90} = 3 > 1$. We conclude that the demand is inelastic when $p = 30$ and elastic when $p = 90$.

13. Let V denote the volume of the soap bubble, and r its radius. Then, we are given $\frac{dV}{dt} = 10$. From the formula, $V = \frac{4}{3}\pi r^3$, we find, upon differentiating with respect to t,

$$\frac{dV}{dt} = \left(\frac{4}{3}\pi\right) 3r^2 \frac{dr}{dt}$$

So $\dfrac{dr}{dt} = \dfrac{dV/dt}{4\pi r^2}$. At the time that $r = 5$, we have

$$\frac{dr}{dt} = \frac{10}{4\pi 5^2} = \frac{1}{10\pi}.$$

So the radius is increasing at the rate of $1/(10\pi)$ cm/sec. On the other hand, from $S = 4\pi r^2$, we can find $\dfrac{ds}{dt} = 4\pi(2r)\dfrac{dr}{dt}$. Therefore, when $r = 5$, we have

$$\frac{ds}{dt} = 8\pi(5)\frac{1}{10\pi} = 4.$$

So, the surface area is increasing at the rate of 4 cm^2/sec.

14.

Values of a	$\lim_{x\to a} F(x)$	Continuous at a?	Differentiable at a?
-3	-2	No. $F(-3)$ is not defined.	No. $F(x)$ is not continuous at -3.
0	-3	Yes.	Yes.
1	DNE. L.H. lim $\neq$ R.H. lim	No. $\lim_{x\to 1} F(x)$ DNE	No. $F(x)$ is not continuous at 1.
2	2	Yes.	No. Graph has sharp corner at $x = 2$
6	-2	Yes.	Yes.
7	DNE. L.H. lim $\neq$ R.H. lim	No. $\lim_{x\to 7} F(x)$ DNE.	No. $F(x)$ is not continuous at 7.

Note 1: A function which is differentiable at $x = a$ must also be continuous at $x = a$, so, since our function is not continuous at $x = -3$, and not continuous at $x = 1$ and not continuous at $x = 7$, it cannot be differentiable at these values of x.

Note 2: "L.H. lim" means left hand limit while "R.H. lim" means right hand limit.