TANGENT PLANES TO SURFACES

Let $\mathrm{f}(x, y)$ be a function of two variables, let (x_0, y_0) be a point in the domain of f, and consider the graph of $z = \mathrm{f}(x, y)$ as a surface in three-dimensional space . We want to answer the question: when will there be a tangent plane to the surface at the point (x_0, y_0, z_0), where $z_0 = \mathrm{f}(x_0, y_0)$, and what are the equations for this tangent plane?

It is easiest first to give the equations for the tangent plane, assuming it exists. We know that, as a plane, it must have an equation of the form

$$z = a\,x + b\,y + c \qquad (1)$$

This plane must contain the point (x_0, y_0, z_0), so $z_0 = a\,x_0 + b\,y_0 + c$. This implies $c = z_0 - a\,x_0 - b\,y_0$, and substituting this in equation (1) for z, leads to $z = a\,x + b\,y + z_0 - a\,x_0 - b\,y_0$. By a rearrangement of terms and by collecting the two terms with the coefficient a and the two terms with the coefficient b, we get

$$z = z_0 + a\,(x - x_0) + b\,(y - y_0)\,. \qquad (2)$$

What are a and b ? For the linear equation (2), recall that the coefficient a is the rate of change of the dependent variable z with respect to variable x when variable y is held fixed. Now consider the first partial derivative $\left(\frac{\partial}{\partial x} f\right)(x_0, y_0)$; it represents the instantaneous rate of change of $z = \mathrm{f}(x, y)$ with respect to x at x_0 when y is held fixed at y_0. The rate of change of z with respect to x when y is held fixed of the tangent plane to $z = \mathrm{f}(x, y)$ should coincide with the instantaneous rate of change of of $z = \mathrm{f}(x, y)$ with respect to x at the point of tangency. Therefore, we should have $a = \left(\frac{\partial}{\partial x} f\right)(x_0, y_0)$. Similar reasoning, but varying y and fixing x, leads to $b = \left(\frac{\partial}{\partial y} f\right)(x_0, y_0)$. Plug these results back into (2), using $z_0 = \mathrm{f}(x_0, y_0)$: the equation for the tangent plane is

$$z = \mathrm{f}(x_0, y_0) + \left(\frac{\partial}{\partial x} f\right)(x_0, y_0)\,(x - x_0) + \left(\frac{\partial}{\partial y} f\right)(x_0, y_0)\,(y - y_0)\,. \qquad (3)$$

Example. Let $\mathrm{f}(x, y) = x^3\,y + x\,y^2$. Find the equation of the tangent plane at $x_0 = 1$, $y_0 = 2$, and $z_0 = \mathrm{f}(1, 2) = 6$.

The first partials of f are:

$$\frac{\partial}{\partial x} f = 3\,x^2\,y + y^2 \qquad \text{and} \qquad \frac{\partial}{\partial y} f = x^3 + 2\,x\,y.$$

Thus $\left(\frac{\partial}{\partial x} f\right)(1,\,2) = 10$ and $\left(\frac{\partial}{\partial y} f\right)(1,\,2) = 5$. By using (3), the equation of the tangent plane is:

$$z = 6 + 10\,(x - 1) + 5\,(y - 2).$$

We have not yet answered the question of when it is reasonable to say that the surface $z = \mathrm{f}(x, y)$ has a tangent plane at $x_0, y_0,\ z_0 = \mathrm{f}(x_0, y_0)$. This is a somewhat technical matter requiring a precise definition of tangent plane. We will not go into it, except to say that it is not enough that the partial derivatives $\frac{\partial}{\partial x} f$ and $\frac{\partial}{\partial y} f$ exist at (x_0, y_0). They must exist at least in a disk in the (x, y)-plane containing (x_0, y_0) in its interior, and they and the function f must be continuous at (x_0, y_0). This will be true for all functions treated in this course when we are interested in tangent planes and so we need not worry about it further.

Remark: Suppose that both partial derivatives are 0 at a point (x_0, y_0). Then the equation of the tangent plane at $x_0, y_0,\ z_0 = (x_0, y_0)$, is $z = \mathrm{f}(x_0, y_0)$, which is just a horizontal plane at level $\mathrm{f}(x_0, y_0)$.