Part I. Integration

Techniques and applications

Double angle formula: $\cos 2x = \cos^2 x - \sin^2 x$

Parts: $\int u\,\mathrm{dv} = uv - \int v\,\mathrm{du}$

Area between curves: $\int_{\text{left}}^{\text{right}} (\text{top} - \text{bottom})\,\mathrm{dx}$.

$\int \sec = \ln|\sec + \tan|$; $\int \frac{1}{1+x^2} = \arctan$; $\int \frac{1}{\sqrt{1-x^2}} = \arcsin$.

Numerical methods

Trapezoidal: $\frac{\Delta x}{2}[1\cdot ? + 2\cdot ? + \ldots 2\cdot ? + 1\cdot ?]$. Error: $\leq \frac{M_2(\Delta x)^2 w}{12}$: M_2 bounds $|f''|$.

Simpson: $\frac{\Delta x}{3}[1\cdot ? + 4\cdot ? + 2\cdot ? \ldots 4\cdot ? + 1\cdot ?]$. Error: $\leq \frac{M_4(\Delta x)^4 w}{180}$: M_4 bounds $|f^{(4)}|$.

(Width $w = |b-a|$)

Taylor polynomial centered at a:

$\sum_i \frac{f^{(i)}(a)}{i!}(x-a)^i$. Remainder $\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ (with c between a and x).

Part II. Matrix Algebra and linear systems

Linear Systems

(Reduced) row echelon form by Gaussian elimination, free variables, basic variables.
Basis of solutions (homogeneous case): set one free variable to 1, the others to 0.
Cramer: $x_i = (\det B_i)/(\det A)$; B_i has the ith column replaced by $\mathbf{b}$.
Invertible case: $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$ in the invertible case.

Determinants

Cofactor expansion: sum $a_{ij}A_{ij}$. Also use row and column operations to simplify.
$\det AB = \det A \det B$; $\det A^T = \det A$.

Inverse:

$1/\det A$ times a matrix whose (i,j)=entry is the cofactor associated with the opposite location (j,i).

Eigenvalues and eigenvectors:

Eigenvalues: roots of the *characteristic polynomial* $\det(A - \lambda I)$.
Eigenvectors $\mathbf{u}$ associated to λ: $A\mathbf{u} = \lambda\mathbf{u}$.
These are found by applying Gaussian elimation to $A - \lambda I$. Dimension: number of independent eigenvectors.

Part III. Leslie Model

Entries. a_i: fertility rates, by cohort (age class), and b_i: survival rates (usually given as mortality rates) by cohort. Serious demographers use *female* mortality rates and count only *female* births, as the fertility rate among males is negligible.

Dominant eigenvalue: rate of growth of population, long-term. Conversion to percentage using 5-year cohorts: $\lambda^5 - 1$.

Eigenvector v_1 (corresponding to λ_1): Long-term population distribution, when normalized to add up to 1 (assuming all age classes are represented).

Perron-Frobenius: under mild conditions, the dominant eigenvalue is positive and has multiplicity 1, and there is a positive eigenvector corresponding to it.

A^n: This can be computed with considerable efficiency by diagonalizing, and with even more efficiency as (approximately) $\lambda_1^n(1/\mathbf{v}_1^T\mathbf{u}_1)\mathbf{v}_1\mathbf{u}_1^T$ where $\mathbf{v}_1$ is the eigenvector for A corresponding to λ_1, and $\mathbf{u}_1$ is the eigenvector for A^T with the same eigenvalue.

Part III. Differential equations

Basics

Separate variables if possible.

Logistic equation

$y' = by(L-y)$, general solution $y = \dfrac{L}{1 \pm Ke^{-at}}$; $(a = bL)$.

Data-fitting:

Evenly spaced times: $t_1 = t_2 - T$, $t_3 = t_2 + T$; $y_i = y(t_i)$; $\bar{y}_i = y_i/y_2$; assume $\bar{y}_1\bar{y}_3 < 1$.

$$K = \frac{(1-\bar{y}_1)(\bar{y}_3 - 1)}{1 - \bar{y}_1\bar{y}_3} \qquad L = y_2(1+K) \qquad a = \frac{1}{T}\ln\left(\frac{Ky_1}{L - y_1}\right)$$

$y = L/(1 + Ke^{-at})$

First order Diff. Eq.

Standard form. $y' + P(x)y = Q(x)$: use $e^{\int P}$ as an *integrating factor*.

Bernoulli type: $y' + P(x)y = Q(x)y^n$; rewrite using $z = y^{1-n}$ (if $n \neq 0, 1$)

2nd order (or higher), constant coefficient

Characteristic polynomial (homogeneous case)

$y = e^{rt}$ or possibly $t^n e^{rt}$; find r. Three cases:

1. Distinct real roots. e^{rt}.
2. Repeated real roots e^{rt}, te^{rt}, ...
3. Complex roots $a \pm bi$: $e^{at}\cos(bt)$, $e^{at}\sin(bt)$.

Inhomogeneous case. Solve the homogeneous case first. Then look for a *special solution* y_p by one of two methods:

1. undetermined coefficients (guess the form of a solution)
2. variation of parameters: $y_p = u_1y_1 + u_2y_2$ and:

$u_1'y_1 + u_2'y_2 = 0$

$u_1'y_1' + u_2'y_2' = F(x)$