

Mathematics 138 Spring 2003, Review Sheet II

1. Use the following statistics for U.S. prison population:

Year	1990	1995	1996	1997
	1,148,702	1,585,586	1,646,020	1,743,643
Year	1998	1999	2000	2001
	1,816,931	1,893,115	1,937,482	1,962,220

(a) Fit the data to a logistic model using the data from 1990, 1995, and 2000. What is the predicted maximum population, and how well does the model fit all known data?

(b) Fit the data to a logistic model using the data from 1995, 1997, and 1999, as in part (a).

(c) Fit the unrestricted growth model to the data, using the last two years shown (2000, 2001).

(d) Make predictions for U.S. prison population 10 years from now, based on the three models. If you were paid by the government to make such predictions, what number would you give as your final answer?

2. A small country has a population of 5 million, a net birth rate of 1% annually, and a net *emigration* rate of 10,000/year.

(a) Express these assumptions as a differential equation for the function $y(t)$ giving the population at time t (in millions), with the current year taken as $t = 0$.

(b) Solve the differential equation and derive an explicit model for the population (a formula for y).

(c) Predict the population 10 years out.

3. Find a curve $y = f(x)$ which is perpendicular to the family of curves $y = C/x$ (C varies): that is, the graph of $f(x)$ is perpendicular to each of these curves wherever they cross. Here are the steps:

(a) Show that each of the given curves $y = C/x$ satisfies the differential equation $y' = -y/x$.

(b) Derive the following differential equation for the unknown curve $y = f(x)$: $y' = x/y$.

(c) Find the *general solution* to the differential equation $y' = x/y$.

(d) Find the three *particular solutions* passing through the following points: (i) $(0, 1)$; (ii) $(1, 1)$; and (iii) $(1, 0)$.

(e) Sketch the curves $y = C/x$ for $C = 0, 1, 4, -1, -4$ on one pair of axes, and sketch the three curves you found in the same picture.

(To avoid distortion, your graphing calculator should use a square window.)

4. Solve the following differential equations with initial conditions $y(0) = 1$, $y'(0) = 2$:

(a) $y'' - 6y' + 5y = 0$

(b) $y'' - 10y' + 25y = 0$

(c) $y'' - 2y' + 5y = 0$

5. A tank containing 20 gallons of pure water has an inflow pipe adding 4 gallons of an alcohol/water mixture (half and half) per minute; the contents of the tank are drained by an outflow pipe at the rate of 3 gallons per minute.

Derive a formula for total alcohol content of the tank after t minutes.

6. Solve the following differential equations with the initial condition $y(0) = 0$:

(a) $y' = e^{x+y}$

(b) $y' = e^x + y$

(c) $y' = e^x y + y^2$

7. Carry out the following calculations:

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1.1 & 0 \\ 0 & .1 \end{bmatrix}^{10}$

8. Calculate the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}$$

Row and column operations will be very helpful.

Answer: 12.

9. Find the inverse of the matrix $\begin{bmatrix} 6 & 14 & -3 \\ -14 & -15 & 4 \\ -3 & -4 & 1 \end{bmatrix}$.

Answer: $\begin{bmatrix} 1 & -2 & 11 \\ 2 & -3 & 18 \\ 11 & -18 & 106 \end{bmatrix}$.

(The determinant, fortunately, turns out to be 1.)

10. Find the eigenvalues, and corresponding eigenvectors, for the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

One of the eigenvectors is $\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$ (or any multiple of this).

11. Suppose A is a square matrix and $A^T A = I$. Show that the determinant of A is ± 1 .