The future value of the income over the time period $T$ is given by $FV = \int_0^T f(t)e^{r(T-t)} dt$.

The present value is given by: $PV = \int_0^T f(t)e^{-rt} dt$

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the $n$th Taylor Polynomial of the function $f$ at $a$.

The remainder Formula: If $|f^{n+1}(x)| \leq M$ to all number between $x$ and $a$, then:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Differential Equations

Euler’s Method: approximates the values of the solutions for the DE $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$ at specific points:

$$y_0 = y(x_0), \quad y_1 = y_0 + hf(x_0, y_0), \quad \ldots \quad y_{n+1} = y_n + hf(x_n, y_n)$$

First order DE: The general solution of a DE of the form $\frac{dy}{dx} + p(x)y = q(x)$ is $y = e^\int p(x)dx$ \[ \frac{1}{I(x)} \left[ \int I(x)q(x)dx + C \right] \]

where $I(x) = e^\int p(x)dx$

Second Order Homogeneous Linear DE; $ay'' + by' + cy = 0$ $a \neq 0$

The characteristic equation of $ay'' + by' + cy = 0$ is $ar^2 + br + c = 0$.

When the CE has 2 distinct real roots, $r_1, r_2$, the solution is $y = C_1e^{r_1x} + C_2e^{r_2x}$.

When the CE has 2 equal real roots, $r_1 = r_2 = r$, the solution is $y = (C_1 + C_2x)e^{rx}$.

When the CE has 2 distinct none real (complex) roots, $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, the solution is $y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$.

Variation Of Parameters: Let $y_h = C_1y_1 + C_2y_2$ be the solution for the homogeneous DE $ay'' + by' + cy = 0$. Then the particular solution for the nonhomogeneous DE $ay'' + by' + cy = F(x)$ is $y_p = uy_1 + vy_2$ where

$$u(x) = \int \frac{-y_2F(x)}{y_1y_2' - y_2y_1'} dx \quad \text{and} \quad v(x) = \int \frac{y_1F(x)}{y_1y_2' - y_2y_1'} dx$$

Note that the solution is $y_h + y_p = C_1y_1 + C_2y_2 + uy_1 + vy_2$
Exponential growth and decay: \( \frac{dQ}{dt} = kQ(t) \).

The Logistics Equation with \( Q_0 < L \):
\[
\frac{dQ}{dt} = aQ - kQ^2 \quad \text{or if let } L = a/k \quad \frac{dQ}{dt} = kQ(L - Q).
\]
The solution of the equation is \( Q(t) = \frac{L}{1 + Ae^{-at}} \) and \( A = \frac{L}{Q_0} - 1 \)
Let \( Q_0 = Q(0) \), \( Q_1 = Q(T) \) and \( Q_2 = Q(2T) \), then:
\[
\frac{1}{Q_1} - \frac{1}{Q_2} = e^{-aT}, \quad \frac{1}{L} = \frac{1}{Q_0} - \frac{A}{L} \text{ and } 1 = \frac{1}{L} Q_0 - A \frac{1}{L}.
\]

Numerical Integration

Trapezoidal Rule: \( \int_a^b f(x)dx \approx T_n = \frac{1}{2} \left( \frac{b-a}{n} \right) [f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)] \)

Simpson’s Rule, \( n \) even: \( \int_a^b f(x)dx \approx S_n = \frac{1}{3} \left( \frac{b-a}{n} \right) [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \)

Trapezoidal Rule Error Bound \( |E_n| \leq \frac{(b-a)^3}{12n^2} M \) when \( |f''(x)| \leq M \) for all \( a \leq x \leq b \).

Simpson’s Rule Error Bound \( |E_n| \leq \frac{K(b-a)^5}{180n^4} \) when \( |f^{(4)}(x)| \leq K \) for all \( a \leq x \leq b \).

Linear Algebra

Matrix \( A \) is invertible only if \( A \) is a square matrix with nonzero determinant. If \( A^{-1} \) exists then \( AA^{-1} = A^{-1}A = I \). The \((i, j)\) entry of \( A^{-1} \) is \( \frac{A_{ji}}{\text{det}A} \) where \( A_{ji} = (-1)^{j+i}M_{ji} \).

Cramer’s Rule: Let \( A \cdot x = b \). Then \( x_i = \frac{\text{det}B_i}{\text{det}A} \), where \( B_i \) is the matrix formed from \( A \) by replacing in the \( i \)th column of \( A \) with the vector \( b \).

Eigenvalues: If \( \lambda \) is an eigenvalue of \( A \) then

the characteristic polynomial of the matrix \( A = \text{det}(A - \lambda I) = 0 \).

\( x \) is an eigenvector of \( A \) for the eigenvalue \( \lambda \) if \( A \cdot x = \lambda x \).

The eigenvectors of \( A \) are also the eigenvectors of \( A^k \) and the eigenvalues of \( A^k \) are \( \lambda^k \)

\[
\int \sin x dx = -\cos x + C \quad \int \cos x dx = \sin x + C \quad \int e^x = e^x + C
\]
\[
\int \frac{1}{x} dx = \ln |x| + C \quad \int x^n = \frac{x^{n+1}}{n+1} + C
\]

Integration by parts: \( \int udv = uv - \int vdu \) and \( \int_a^b uv \|_{a}^{b} = \int_a^b vdu - \int_a^b vdu \)