One-sided limits

A modification of the idea of limit, denoted

$$\lim_{x \to a^-} f(x) = L$$

considers only values of $x < a$. Similarly,

$$\lim_{x \to a^+} f(x) = L$$

considers only values of $x > a$.

If the limit exists, then both one-sided limits also exist and are equal to the limit.

Functions defined by cases

One-sided limits are particularly useful for functions like

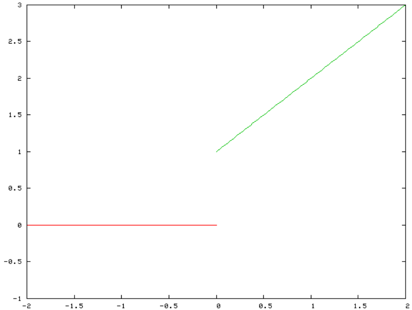
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x + 1 & \text{if } x > 0 \end{cases}.$$

If $x < 0$, then $f(x)$ is indistinguishable from the constant function 0. Since constant functions are continuous, the limit as $x \to 0^-$ of $f(x)$ is 0. If $x > 0$, then $f(x)$ is indistinguishable from $x + 1$ whose limit as $x \to 0^+$ is 1 (again because of continuity).

If f had a limit as $x \to 0$, these one-sided limits would be the same. They **aren't**, so f has no limit as $x \to 0$.

The graph

Here is the graph of f from the previous slide:



Jump discontinuities

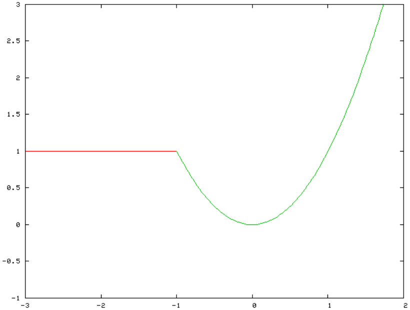
The two **different** one-sided limits should be **visible**. The **jump** at $x = 0$ is a common type of **discontinuity** — however you define the function there. The inability to make the function continuous by redefining it at a point says that the limit at that point **does not exist**.

A continuous function defined by cases

Consider the function the function

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ x^2 & \text{if } x > -1 \end{cases}.$$

Its graph



This function has a limit

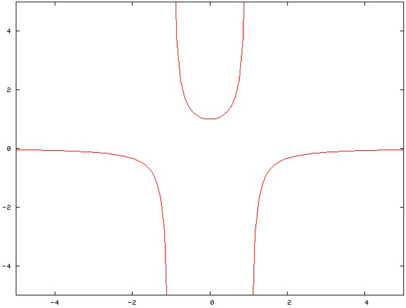
As long as the expressions in each case are continuous, limits can be found **by evaluation**. At a point where two intervals meet, the formula on the interval to the left of the point is used to find one one-sided limit and the formula on the interval to the right is used for the other. If these limits are the same, that value can be shown to be the limit. For $g(x)$, the limit as $x \to -1$ is 1.

It’s not always this easy

Any property of real numbers (e.g., being **rational**) can be used in definition by cases. Functions defined on intervals are just easier to draw, so they are seem more familiar. The principle is the same: as long as one case contains points arbitrarily close to a, the existence of a limit as $x \to a$ implies that the limit restricted to case must equal the unrestricted limit. For the limit to exist, all restricted limits must exist and be equal. For any partition into finitely many cases, the converse is also true.

Another graph

Here is the graph of $y = (1 - x^2)^{-1}$



Infinite limits

This graph seems to leave the **graphing window** through the top and bottom edges in an orderly fashion. This is due to the denominator being zero at $x = \pm 1$. It is often useful to treat $\pm\infty$ as numbers at the far ends of the number line and say that (usually one-sided) limits for which the graph exits at the top have limit $+\infty$ and those exiting at the bottom have limit $-\infty$.

An arbitrary definition

They main feature of an **infinite limit** is that the function **takes no sensible values** close to the point of interest. The textbook uses the symbol ∞ to mean $+\infty$ so it requires a graph to exit through the top of the graphing window. This often happens only on one side of the point of interest, so **one-sided** limits are used in this discussion. If there weren't such a clear picture, this would be an excessively technical definition.

Vertical asymptotes

The graph of a function f near a point where f has an infinite limit has a **vertical asymptote**. The function is defined near this point, so the graph gets arbitrarily close to the line with this value of x, but **only** for y of large absolute value. Any finite value of the function at this point wouldn't look right, so the function is usually left undefined and the graph approaches the "ends" of line.

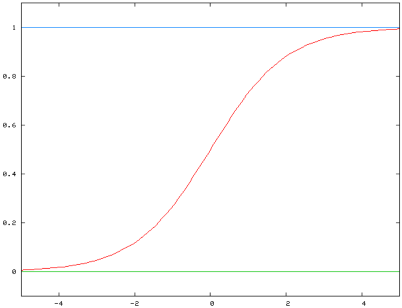
Horizontal asymptotes

The graph of $y = (1 - x^2)^{-1}$ is close to the x-axis (i.e., the line $y = 0$) if $|x|$ is large. The line $y = 0$ is said to be a **horizontal asymptote** in this case. In general, a graph is said to have $y = b$ as a horizontal asymptote if it always leaves a large graphing window near this value of y on **one** of the vertical sides of the window. A formal expression of this uses limits as $x \to +\infty$ of as $x \to -\infty$. Precise definitions of such limits can be given.

Different left and right asymptotes

A rational function always behaves similarly at $\pm\infty$, but other functions can have different limits (and hence different horizontal asymptotes) on the far left or far right of the number line. A simple example is $\left(1 + e^{-x}\right)^{-1}$, shown on the next slide with its horizontal asymptotes at $y = 0$ and $y = 1$.

The graph



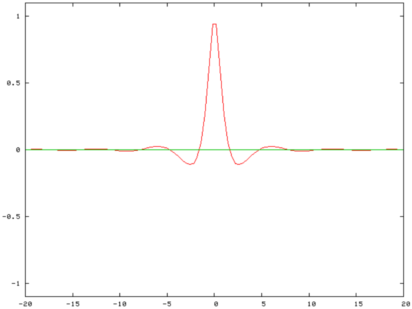
Curves can cross horizontal asymptotes

The definition of **function** forces the graph of a function to approach a vertical asymptote without crossing it. There is no such requirement for **other** asymptotes. The only requirement is that the function be close to the line as $x \to +\infty$ or as $x \to -\infty$. The next slide shows

$$y = \frac{\cos x}{1 + x^2}$$

with its asymptote $y = 0$.

The graph



Finding limits: algebraic trick 1

The example

$$\lim_{x \to 0} \frac{x}{x} = 1$$

generalizes to apply to

$$\lim_{x \to a} \frac{p(x)}{q(x)} = 1$$

for any polynomials p and q whenever $p(a) = q(a) = 0$ because the given condition assures that there is a factor of $x - a$ in both $p(x)$ and $q(x)$. Removing this **common factor** leaves a simpler expression. Usually, there isn't a **second** factor of $x - a$ in $q(x)$, so the limit can now be found by evaluation.

Finding limits: algebraic trick 2

To find

$$\lim_{x \to \infty} \frac{p(x)}{q(x)}$$

for polynomials p and q, write $x = 1/u$ and **simplify**.

A refined version of this is that a polynomial behaves like its **leading term** as the variable goes to ∞, so a rational function behaves like the quotient of leading terms.

Finding limits: algebraic trick 3

To evaluate

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = 1$$

rationalize the **numerator** by multiplying numerator and denominator by $\sqrt{x} + 1$