Tangent lines

The tangent line to the graph of $y = f(x)$ has been defined as the line approximated by **secant lines** joining $\big(a, f(a)\big)$ to a **nearby point on the graph**.

All lines through this point have the form

$$y - f(a) = m * (x - a)$$

(an explicit multiplication symbol $*$ is shown to emphasize that m is a constant factor and not the name of a function being applied at $x - a$)

Slopes and secants

The number m is called the **slope** of the line. If we are interested in the secant line that also passes through $\left(b, f(b)\right)$, then

$$m = \frac{f(b) - f(a)}{b - a}$$

If the point $\left(a, f(a)\right)$ is fixed, then lines through this point are completely described by the value of m, and **nearby** lines have **nearby** values of m.

Approaching the tangent

Putting this together with the idea of **limit** gives that the tangent line to $y = f(x)$ at $\big(a, f(a)\big)$ has the equation $y - f(a) = m * (x - a)$ with

$$m = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

An example

Take $f(x) = x^2$ and $a = 1$. Then

$$m = \lim_{b \to 1} \frac{b^2 - 1}{b - 1}$$

Division of polynomials tells us that

$$\frac{b^2 - 1}{b - 1} = b + 1$$

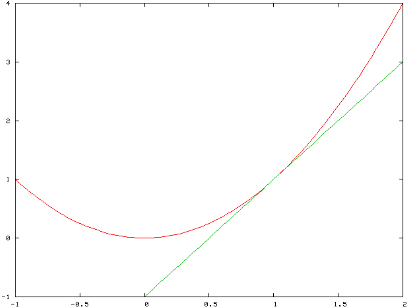
as long as $b \neq 1$.

An example, page 2

Since we **never** consider the point at which we are finding the limit, these expressions are equal wherever we need to calculate them. The continuity of the function taking b to $b + 1$ says that the limit can be found by evaluating this expression at $b = 1$ to get 2.

Verification

Here is a graph of the given curve and the computed tangent line



It looks right!

A technical change

The algebra of finding an equivalent form of the **difference quotient** can often be simplified by writing $b = a + h$ and considering the limit as $h \to 0$. This has the effect of making the denominator always be h. For the previous example, the numerator is

$$(1+h)^2 - 1 = (1 + 2h + h^2) - 1 = 2h + h^2$$

After replacing h/h by 1, the quotient becomes $2 + h$, whose limit is 2.

Exercises 2.7

5: $f(x) = x^2 + 2x$, $a = -3$

6: $f(x) = x^3$, $a = -1$

9: $f(x) = \dfrac{x-1}{x-2}$, $a = 3$

Velocity

The problem of finding **instantaneous velocity** from a description of distance as a function of time has been shown to be equivalent to the problem of finding the slope of a tangent line. Any method that solves one problem can be used to solved the other. The **words** change, but the **mathematics** is the same.

Other settings involving **rates of change** are given in the textbook. They all involve the same mathematics.

The derivative

Section 2.8 begins with the definition: given a function f and a number a,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is called the **derivative** of f **at** a. This is denoted (using a notation whose significance is not explained until the next section) $f'(a)$.

In order to use this notation, you need a simple name for the function that you can follow with $'$ to get a name for the derivative.

Exercises 2.8

Here, derivatives are found for all a.

13: $f(x) = 3 - 2x + 4x^2$

16: $f(x) = \dfrac{x^2 + 1}{x - 2}$

17: $f(x) = \dfrac{1}{\sqrt{x + 2}}$

The derivative as a function

The calculation of the derivative is a process that, for each given number a yields a value that is the slope of the tangent line to $y = f(x)$ at the point where $x = a$ (or whatever your favorite version of the derivative is). This process results in **at most one answer**. That is, it defines a function. We have already hinted in our notation that the name of this function is f'.

In fact, the **process** for finding $f'(a)$ often allows a to be a parameter rather than requiring that a numerical value of a be given before finding the derivative.

Calculus and its applications

We haven't really done any calculus yet! In calculus, f' is found by direct manipulation with an expression defining f. Limits are used to prove theorems **justifying the rules** of calculus. In calculus, the derivative is found **as a function** before finding any of its values.

However, **all applications of the derivative use the value of the derivative at a particular point**. In particular, to find the tangent line to $y = f(x)$ at the point where $x = a$, you do: (1) find $f'(x)$ by calculus; (2) evaluate $f(a)$ and $f'(a)$; (3) put these **numbers** into $y - f(a) = f'(a) * (x - a)$; (4) simplify, if necessary.

Other notation

If we are able to write $y = f(x)$, then

$$f'(x) \text{ may be written } \frac{dy}{dx}$$

Other notation is mentioned in the text. I try not to use $\frac{df}{dx}$ since this blurs the distinction between functions and the expressions used to define them. My rule is that differentiation of functions is an **abstract** property of the function that must not depend on using a particular variable in the definition of the function. Notation using the name of a variable should apply only to expressions or variables representing expressions.

Differentiability implies continuity

A function f is said to be **differentiable** if you can find f'. Calculus is based on theorems that prove differentiability by a method that gives a formula for the derivative. Thus, in practice, you can easily convince yourself that certain functions are differentiable. Theorem 4 on page 171 prove that a function that is differentiable at a point is continuous at that point by looking at the limit as $x \to a$ of

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

Proofs of continuity

This general argument can be used to produce a direct verification of the ϵ-δ definition of continuity for any differentiable function. Here is a sketch of the method.

Since

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

the difference quotient is close to $f'(a)$ if x is close to a. A weak consequence of this is that the difference quotient is bounded on some interval containing a in its interior. It is often easy to produce a fairly large interval with this property.

Proofs of continuity, page 2

If the absolute value of the difference quotient is at most M on an interval I, then taking $|x - a| < \epsilon/M$ will force $|f(x) - f(a)| < \epsilon$. This is almost what we need — we also need to assure that $x \in I$. The expression for δ is the minimum of the values needed for these two conditions. Verifying continuity from the definition is easy once you have this technique, but you need to have met the derivative to understand it.