# Rates of change

Note every use of the word “rate” refers to a **rate of change**, which is expressed by a derivative, but there are many examples. Many subjects have developed special techniques for using the results of calculus for processing their rates. It is not the job of this course to explore those special techniques; rather, we will give general principles to recognize derivative where they are needed and to **do** the required calculations with derivatives.

# The whole story is more complicated

You should not expect to completely understand all applications of the derivative based on having been introduced to the definition. In many cases, derivatives are used to reformulate problems as **differential equations**, so a complete understanding requires methods for working with, and sometimes solving, such equations. Except for a few simple examples, those skills only appear in the fourth semester of a calculus sequence. Section 3.3 is only the "Call me Ishmael" of *Moby Dick*.

# Exercise 3.3#9

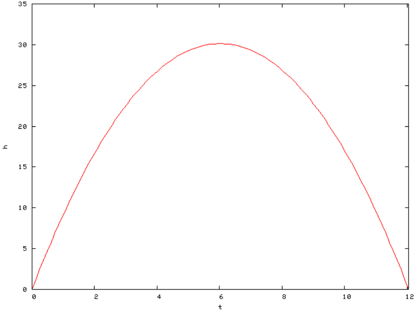
If a stone is thrown vertically upward from the surface of the moon with a velocity of $10m/s$, its height (in meters) after $t$ seconds is $h = 10t - 0.83t^2$.

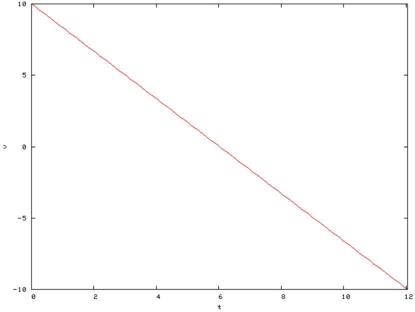
(a) What is the velocity of the stone after 3 seconds?

(b) What is the velocity of the stone after it has risen 25 meters?

# Two graphs

Velocity is the derivative of distance, so $v = 10 - 1.66t$. Here are the graphs of $h$ and $v$ as functions of $t$:




$h$ left $\qquad$ $v$ right

# Exercise 3.3#29

Suppose that he cost (in dollars) for a company to produce $x$ pairs of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

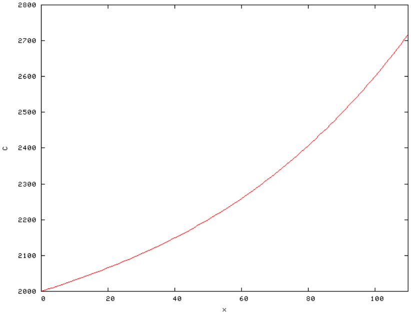
(a) Find the marginal cost function.

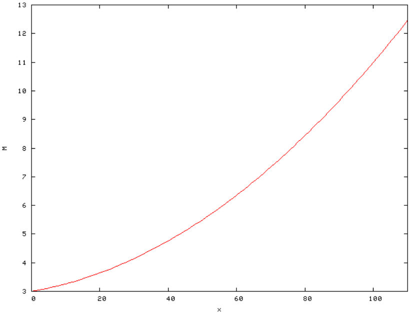
(b) Find $C'(100)$ and explain its meaning. What does it predict?

(c) Compare $C'(100)$ with the cost of producing the $101^{\text{st}}$ pair of jeans.

# Two graphs

Marginal cost $M$ s the derivative of cost $C$, so $M = 3 + 0.02x + 0.0006x^2$. Here are the graphs of $C$ and $M$ as functions of $x$:




$C$ left

$M$ right

# Derivatives of trig functions

To differentiate a **new** function, you must appeal to the definition. The derivative of $f(x) = \sin x$ will depend on the **addition formula** for $\sin x$ and the two limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

We will not be able to **prove** the first limit statement, but we will show that it is plausible. The second limit statement will be shown to be a consequence of the first.

# The addition formula

Step 1 of the process for finding $f'(x)$ is to find $f(x+h)$. For $f(x) = \sin x$, this is

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

Step 2 then gives $f(x+h) - f(x)$ as

$$\sin x \left( \cos h - 1 \right) + \cos x \sin h$$

# Using the limits

Dividing these terms by $h$ and using the claimed values of the limits gives the derivative $f'(x) = \cos x$.

The proof of the second limit formula uses the fundamental identity

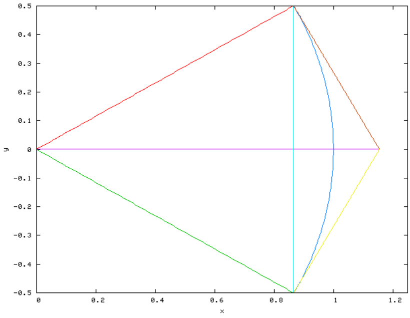
$$\cos^2 x + \sin^2 x = 1$$

and trick of **rationalizing the numerator**.

$$\cos x - 1 = \frac{\cos^2 x - 1}{\cos x + 1} = \frac{-\sin^2 x}{\cos x + 1}$$

# The new limit (or "why radians?")

Here is the picture that contains the ideas of the proof.



# Determining the limit (or not)

As $h \to 0$, the inscribed and circumscribed figures get closer to the circular sector and all measurements (either area or perimeter) can be found. Calling this a "proof" has been criticized as **circular reasoning** since calculus (including the formula for the derivative of $\sin x$) is used to determine the area or perimeter of a circular sector.

# How Archimedes found pi

Imagine $2^n$ of these figures with angles $\pi/2^n$ about the central line fit together to fill a circle with inscribed and circumscribed polygons. The area of the whole circle is trapped between the areas of two polygons. Although formulas for the areas of these polygons require trig functions of many angles, there is an **algebraic** way to relate the areas in figures for two consecutive values of $n$. This leads to a calculation the gives better and better approximations to $\pi$. Archimedes did this when these calculations were much harder to do than they are now.

# What does “co” mean

Every trig function has a **cofunction**. Instead of giving three separate definitions of these functions, it is useful to note the “co” is short for **complement**, and the complement of an angle $\theta$ is the angles $\pi/2 - \theta$. We shall see that the derivative of a cofunction is **the negative of the cofunction** of the derivative of the function. Also, a co-co-function is the original function. Thus the derivative of $\cos x$ is $-\sin x$.

# Derivative of the tangent

Since $\tan x = \sin x / \cos x$, the quotient rule gives

$$\frac{d}{dx}\tan x = \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

# Derivative of the secant

Since $\sec x = 1/\cos x$, the quotient rule gives

$$\frac{d}{dx}\sec x = \frac{-\frac{d}{dx}\cos x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$