Inverse functions

A special case of an implicit definition of a function is the case of an **inverse function**. In Leibniz form, the question is: if y depends on x and you know dy/dx, how does x depend on y and what is dx/dy?

The short answer is the plausible statement

$$\frac{dx}{dy} = 1 \Bigg/ \frac{dy}{dx}$$

What does that mean?

Here y starts out as a function of x, so dy/dx and its reciprocal are easily obtained in terms of x, but we want dx/dy in terms of y. To obtain this, every x must be replaced by its equivalent as a function of y, or other information used to obtain the expressions we have in terms of y.

A deferred example

The best example of a pair of inverse functions are the exponential and logarithm, but these are so important that they are kept to themselves in section 3.8.

The example treated in section 3.6 will be done next.

Arcsine

If $y = \sin x$, then $dy/dx = \cos x$. Since $\sin^2 x + \cos^2 x = 1$,

$$\cos x = \pm\sqrt{1 - \sin^2 x}.$$

Actually, the restriction to $-\pi/2 \le x \le \pi/2$ when $x = \arcsin y$ shows that the $+$ sign always holds. Thus

$$\frac{dx}{dy} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}$$

Arctangent

If $y = \tan x$, then $dy/dx = \sec^2 x = 1 + \tan^2 x$. Thus

$$\frac{dx}{dy} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}$$

Higher derivatives

The methods of Calculus lead directly to an expression for the derivative of a function of x as a function of x.

When we have such a function, we can differentiate it. All that we need is a notation. With functions, we simply add another prime: $f''(x)$ is the derivative of $f'(x)$. The Leibniz form is more awkward:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \text{is written} \quad \frac{d^2y}{dx^2}$$