Indeterminate forms

In almost all cases of

$$\lim_{x \to a} f(x),$$

$f(x)$ is a continuous function at $x = a$, so the limit is $f(a)$, calculated by **plugging in** a for x and **simplifying** the expression.

This is so **routine**, that is the **exceptions** that are **emphasized**. Exceptions for which a **reasonable limit** is likely to exist although the formula cannot be evaluated, are those where $f(a)$ has the form

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \text{or} \quad \infty - \infty$$

This is related to the derivative

The $0/0$ expression has been met in the definition of the derivative. The numerator, $f(x+h) - f(x)$ is zero when $h = 0$, and the denominator, h, is also zero when $h = 0$. In many examples, $f(x+h) - f(x)$ can be reduced by **algebra** to exhibit a factor of h. For $h \neq 0$, the difference quotient simplifies to an **equivalent expression** that does have a value at $h = 0$. Evaluation of the limit only requires **identifying** the equivalent expression.

Recognizing a derivative

The examples

$$\frac{\sin x}{x} \quad \frac{e^x - 1}{x} \quad \frac{\ln(1+x)}{x}$$

as $x \to 0$ have already been mentioned as examples in which the limit is recognized as a special case of a known differentiation formula.

An extension of this method

Suppose that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

and $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f_1(x)}{g_1(x)}$$

where

$$f_1(x) = \frac{f(x) - f(a)}{x - a}$$

and similarly for g_1. The limits of $f_1(x)$ and $g_1(x)$ are $f'(a)$ and $g'(a)$ and the quotient of these can be found.

l'Hôpital's rule

We have just proved that, if f and g are continuous functions with continuous derivatives and $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

This form is good enough for many of the exercises, but a more satisfying version omits the pale phrases. This result is true, and has the name at the top of the page, but it is harder to prove.

Extending l'Hôpital's rule

The same principle also applies to the ∞/∞ **indeterminate form**, although different ideas are required for the proof. Practice is necessary to identify the form of the rule that should be used to evaluate a particular limit.

Additional indeterminate forms like $0 \cdot \infty$, 1^∞, ∞^0, and 0^0 are modifications of the indeterminate forms mentioned earlier. They are analyzed by identifying the **fraction** to which l'Hôpital's rule applies.

Exercises

The use of these rules will be illustrated with the following exercises from Section 4.4:

5, 7, 9, 18, 23, 25, 26, 43, 45

Exercise 63

$$\lim_{x\to\infty} x\left(\ln(x+5) - \ln x\right)$$

The limit will be calculated at the blackboard. A graph is included here.