Visualizing the first derivative

The **value** of the first derivative **at a point** is the slope of the tangent line at the point. Derivative zero means a horizontal tangent. Positive derivatives mean that the tangent line is the graph of an increasing function; **large** positive derivatives mean that the tangent line is the graph of a **rapidly** increasing function.

Similarly, negative derivatives mean that the tangent line is the graph of a decreasing function.

Properties of the curve itself

The Mean Value theorem tells us that, if the derivative is positive on an interval, then the function is increasing on the interval, and that if the derivative is negative on an interval, then the function is decreasing on that interval. If the derivative is continuous, a change of sign requires that the derivative be zero somewhere on an interval where the sign changes. Since most of the functions that you know are rarely zero, locating the zeroes of the derivative separates the domain of the functions into intervals on which the function is increasing and intervals where it is decreasing.

Examples

Consider the functions given by the expressions

$$x^2 \quad \text{and} \quad \sin x$$

The key features of the graphs of these functions are the intervals where the function increases or decreases and the points with horizontal tangents separating them.

Which way is up?

That is, how do we distinguish minima from maxima? The simple example x^2 summarizes the distinction: for $x < 0$, the **difference quotient**

$$\frac{f(x) - f(0)}{x - 0}$$

is equal to the value of the derivative **somewhere** between x and 0 (you can find **where**, but that isn't relevant). This value is negative. Similarly, the derivative is positive for $x > 0$. The conclusion: **at a minimum, the derivative changes from negative to positive**.

The second derivative test

A function that changes from negative to positive at a point must be **increasing** at that point.

A function $f(x)$ has a maximum at $x = a$ if and only if the function $-f(x)$ has a minimum at $x = a$, so the **statement** of the previous result gives the following result.

The second derivative test, part 2

A function that changes from positive to negative at a point must be **decreasing** at that point.

Withe f equal to **the derivative** of the function being considered, we have that a function has a local minimum if its first derivative is zero and its second derivative is positive.

This is a derived property

The converse isn't quite true. The function $f(x) = x^4$ has a minimum at $x = 0$ although its second derivative at $x = 0$ is zero (instead of being positive). It is easy to check the sign of the derivative in this case, so the nature of the critical value could be determined without computing the second derivative, and computing the second derivative doesn't help reach that conclusion.

Similarly, if you can show that the function has a **global** minimum at a point, that point must also be a **local** minimum.

Concavity

At a local minimum point, a curve lies above it horizontal tangent. This property may be generalized to apply to arbitrary points.

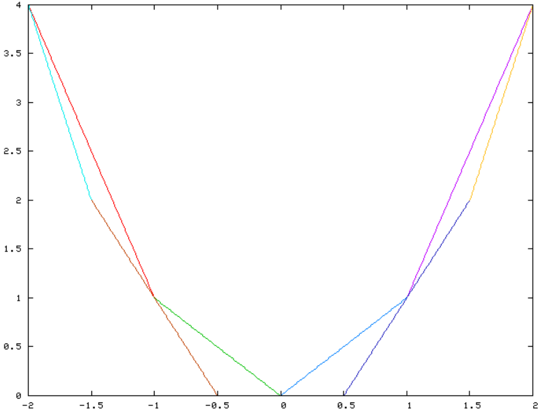
A graph is said to be **concave upward** on an interval if it lies above all of its tangent lines at points of the interval.

This essentially says that the second derivative of the function defining the graph is positive.

Chords

If a curve is concave upward on an interval, it lies **below** all of its chords. This is proved using the same ideas connecting the definition to the characterization by the sign of the second derivative. The next slide shows the bounds on $y = x^2$ determined by the values at 0, ± 1, and ± 2

Bounds on the parabola



Example from end of lecture

The need to revise these notes allows the study of 4.3#37 to be added. We had

$$f(x) = 3x^5 - 5x^3 + 3$$

$$f'(x) = 15x^4 - 15x^2$$

$$f''(x) = 60x^3 - 30x$$

Important points on the graph

It is difficult to see anything about the roots of $f(x)$, but $f'(x)$ and $f''(x)$ are easily factored giving critical values at $x = 0$ and $x = \pm 1$. The **possible** inflection points are at $x = 0$ and $x = \pm 1/\sqrt{2}$. A little experimentation shows that a reasonable graphing window should have $-1.5 < x < 1.5$ These values will be used to find upper an lower bounds using the methods described in earlier slides. A graph of $y = f(x)$ is included to relate the bounds to the result we want to produce.

The graph