Inverting the chain rule

Suppose that $z = F(y)$ and $y = G(x)$. Then, the chain rule says

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

If we write $F' = f$ and $G' = g$, then

$$\frac{dz}{dx} = f(y)g(x) = f\big(G(x)\big)g(x)$$

Then, z as a function of x is

$$\int f\big(G(x)\big)g(x)\,dx$$

A formal version

If we agree that y is **always** $G(x)$, and z is **always** $F(y)$, then $dy = g(x)\,dx$ and

$$z = \int f\big(G(x)\big)g(x)\,dx = \int f(y)\,dy = \int f\big(G(x)\big)g(x)\,dx.$$

The first integral gives z in terms of y, and we express the result in terms of x by using $y = G(x)$. The second integral tells how to obtain his result by integrating **something** with respect to x. These two are **formally** the same if

$$f(y) = f\big(G(x)\big) \quad \text{and} \quad dy = g(x)\,dx$$

Isn't that nice!

Forwards and backwards

There are two ways to use this **substitution rule**. One is to start from $\int f(y)\,dy$ and replace y by $G(x)$ and dy by $g(x)\,dx$. No such examples appear in the current section, but the trigonometric substitutions of section 7.3 are of this type. The other type of substitution is **recognize** that **most** of the integrand is of the form $f\big(G(x)\big)$, and **the rest** is a factor of $g(x)$.

Definite integrals

The textbook gives the example

$$\int_0^4 \sqrt{2x+1}\,dx$$

with $y = 2x + 1$ and $dy = 2\,dx$. The **indefinite** integral is

$$\int \sqrt{2x+1}\,dx = \int y^{1/2} \cdot \frac{1}{2}\,dy = \frac{1}{3} y^{3/2}$$

When evaluating the **definite** integral, this function need only be evaluated a $x = 0$ and $x = 4$, and for these x, we have $y = 1$ and $y = 9$. Instead of replacing y be the **general** expression in therms of x, we can find the numerical values at the ends of the interval.

The formula for definite integrals

In this example, we have

$$\int_{x=0}^{x=4} \sqrt{2x+1}\,dx = \int_{y=1}^{y=9} \frac{1}{2} y^{1/2}\,dy =$$

$$\left[\frac{1}{3} y^{3/2}\right]_{y=1}^{y=9} = \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{26}{3}$$

To clarify how this evaluation is done, we have added explicit mention of the variable taking the particular numerical values.

Exercises 5.5

1. $\int \cos 3x \, dx$

4. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx$

9. $\int (3x - 2)^{20} \, dx$

11. $\int \frac{(1 + 4x) \, dx}{\sqrt{1 + x + 2x^2}}$

26. $\int (1 + \tan \theta)^5 \sec^2 \theta \, d\theta$

More Exercises 5.5

31. $\displaystyle\int \frac{dx}{x \ln x}$

42. $\displaystyle\int \frac{x\,dx}{1 + x^4}$

58. $\displaystyle\int_0^1 x e^{-x^2}\,dx$

62. $\displaystyle\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$