Math 152:10–12 — Fall 1999
TF3 CHM-201
Prof. Bumby

Parametric Equations. Theoretical introductions to the methods of calculus are always given in terms of *functions*. In the first examples, these functions are always given explicitly in using the functions that appear on your calculator's keypad: exponential and logarithm, trigonometric functions and their inverses, and functions built from these by addition, subtraction, multiplication, division, and composition of functions.

The main shortcoming of this approach is that it has always been difficult to apply these methods to what was once considered the most perfect curve of all — the circle. Although, we have the simple equation $(x-a)^2 + (y-b)^2 = r^2$ that characterizes points on a circle of radius r with center (a, b), the equation has difficulty being expressed using functions. This is a fundamental difficulty, because lines usually meet a circle in two points. while the definition of *function* requires that a line can meet the graph of a function it at most one point.

In differential calculus we got around this by inventing the idea of *implicit* function. The idea is that in differential calculus, you only need to consider a small piece of a curve at one time, and — unless you are very unlucky — this piece is the graph of a function of one of the coordinates. This works because the theory is based on functions, but its implementation can use any expression relating variables. This example has a very encouraging property: it gives an easy way to obtain within calculus a statement whose interpretation is the fundamental geometric fact that the tangent to a circle at a point is perpendicular to the line joining that point to the center of the circle.

However, we still don't have a useful way to describe circles in the integral calculus. To find the area of a circle, we are led to an integral that is evaluated using a trigonometric substitution. The perimeter of a circle is more troublesome. The most we can do at one time is a semicircle, and that integral is improper.

To remedy this, we give a description of curves that is suitable for integral calculus. The idea is that the curve is not just a mark that you have found on a piece of paper — it was *drawn*. The rules for drawing a curve require that you know where your writing instrument is at all times. That is, you want the coordinates x and y to be functions of t. This will be expressed by saying that curve is given in terms of a *parameter* t. Different ways of drawing the same curve lead to the parameter in one case being a function of the parameter in the other. If a quantity is given by an integral, a substitution using this function will allow the calculations based on different parameterizations to be compared. The quantities with geometric significance will turn out to be independent of the way the parameterization. The familiar formulas like

$$A = \int y\,dx \text{ and } s = \int \sqrt{dx^2 + dy^2}$$

lead directly to integrals in terms of parameters. This notation allows you to move quickly to these integrals without retracing their theoretical justification. For the circle, the usual parameterization includes the substitution that simplifies the calculation of the integrals giving length of circular arcs or areas of figures bounded by them.

Some new curves. Curves called *cycloids* obtained by tracing the motion of a point on a circle rolling on a line or another circle. have nice parametric descriptions, although it is awkward to eliminate the parameter to get a relation between x and y for these curves. These curves have appeared as solutions of important problems in addition to appearing as a misguided attempt to describe the motion of the planets.

Polar coordinates. If you were to try to describe a curve using the angle θ from some reference direction as a parameter, the first step to recovering the rectangular coordinates might be to determine the distance r from the origin as a function of θ. If the reference direction if the positive x axis, then

$$x = r\cos\theta \text{ and } y = r\sin\theta \tag{P}$$

These equations describe the polar coordinate system. It has other uses aside from an intermediate step in using θ as a parameter for describing some curves, but true two dimensional uses of polar coordinates will depend on having a two dimensional calculus, which is developed in the third semester course.

If you are given rectangular coordinates and asked to find polar coordinates, you would probably begin by setting $r = \sqrt{x^2 + y^2}$, and then θ would be set to the unique angle between 0 and 2π whose cosine is x/r and whose sine is y/r. That is, θ is chosen to face the point $(x/r, y/r)$ on the unit circle. If you are in a hurry, you might think that θ can be found from $\tan\theta = y/x$. However, the arctangent only takes values between $-\pi/2$ and $+\pi/2$, and it only determines the direction of the line containing the point (x, y). Sometimes, (x, y) is behind you when you face in the direction given by the arctangent. Thus, although (P) gives x and y as a function of r and θ, every attempt to find an inverse function will break down as soon as you find a good reason to assign two different polar coordinates to the same point. In fact, given $(x, y) \neq (0, 0)$, equation (P) has infinitely many solutions. Not only can you add 2π to θ while keeping the same r, but you can add π to θ and change the sign of r. One of the main properties of curves defined in polar coordinates is the angle which, when added to θ brings you back to the same point. A point will lie on a curve if *some* polar description of the point satisfies the equation of the curve. This adds an extra level of difficulty to problems like determining points of intersection of two curves. Although you may be able to see points of intersection by sketching the graph, an exact determination may be elusive. If $r = 0$ in (P), the point is the origin independent of θ. If the polar coordinates $(0, \theta)$ satisfy the equation of a curve, the origin lies on the curve and the value of θ gives the direction of a tangent line at the origin.

More new curves. A family of curve called *limaçons*, with equations like

$$r = A + B\cos\theta,$$

are the main examples of curves having nice polar equations. Figure 18 on p. 551 shows several curves in this family.

Another family of curves that will appear in examples are the *roses*

$$r = \cos k\theta$$

for various values of k. These curves pass through the origin whenever $k\theta$ is an odd multiple of $\pi/2$.

Details of this segment of the course. There is a rearrangement of the schedule because of the Thanksgiving Recess in which Friday classes meet on Wednesday, November 24, so we will have two lectures that week, but no workshop class. This work leads to an exam on Friday, December 03.

Date	Section	Page	Discussion Problems	**Hand-in**
November 12	9.1	531	2, 6, 10	**12**
November 16	9.2	538	2, 4, 6, 10, 16, 30	**32**
November 19	9.3	543	2, 8, 16	**6**
November 23	9.4	552	8, 12, 26, 34, 40, 50, 54	**28**
November 24	9.5	558	6, 12, 18, 20, 24, 44, 48	**8**
November 30	9.R	571	4, 8, 20, 28	**22**

Problems from sections 9.1 and 9.2 should be prepared for the workshop class on November 18; the hand-in problems from these sections should be submitted for grading. The remaining sections should be prepared for the workshop class on December 02.