Trigonometric integrals. Our aim is to show that every integral of the form

$$\int \sin^m x \cos^n x \, dx \qquad (E)$$

where m and n are integers (positive, negative or zero) can be integrated in terms of known functions. The traditional treatment of this is complicated by the bad habit of treating negative powers of $\sin x$ and $\cos x$ as positive powers of different functions — $\csc x$ and $\sec x$, respectively. To further confuse matters, the exponent -1 used to denote the inverse function instead of the multiplicative inverse.

The treatment in the textbook builds on its earlier development of the differential calculus. This leads to separate formulas for three cases of formula (E).

Case I: If $m \geq 0$ and $n \geq 0$, the integral is left in terms of $\sin x$ and $\cos x$, and split into two sub-cases.

> Case IA: If $m + n$ is odd, the integral can be expressed as a polynomial in either $\sin x$ or $\cos x$.

> Case IB: If $m + n$ is even, the integrand can be expressed as a polynomial in $\sin 2x$ and $\cos 2x$, so the methods of Case I can be applied with smaller exponents.

The case $m = n = 0$ is included in Case I. You know what that integral is, be it is different from the type of terms produced in Case IA.

Case II: If $m \geq 0$, $n < 0$, and $m + n \leq 0$, then

$$\sin^m x \cos^n x = \tan^m x \sec^{-m-n} x$$

and the exponents on the right are nonnegative. These can be integrated by simple identities, as in Case IA if the exponent of $\sec x$ is even or the exponent of $\tan x$ is odd. For the remaining cases, it is suggested that integration by parts be used to reduce to the special formula

$$\int \sec x \, dx = \ln \left| \sec x + \tan x \right| + C. \qquad (S)$$

Although a “derivation” of this formula is often presented (see Example 8 of Section 7.2, p. 426), the *real*

proof of this formula is to be lucky enough to have once differentiated the right side of equation (S), simplified the result, and remembered the answer.

Case III: If $m < 0$, $n \geq 0$, and $m + n \leq 0$, then the term can be written using positive powers of $\cot x$ and $\csc x$. Details are usually omitted because the formulas involve no ideas not met in Case II.

By relating the formulas in the text to the basic formula (E), we see that we are a long way from the claim that all such integrals are known. However, the missing cases can be treated by using some form of $\sin^2 x + \cos^2 x = 1$, possibly several times. Details will be skipped for two reasons. First, a uniform proof will be sketched below. Second, the only problems of this type you will meet in the course will allow solution be the methods featured in the textbook.

An all-purpose reduction formula. We differentiate $\sin^a x \cos^b x$ and rewrite the answer in several ways using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$. Each of these says that the derivative can be expressed as a sum of two terms of the same type. Integrating,

we get that $\sin^a x \cos^b x$ is a sum of constant multiples of two integrals of the form (E). In other words, one integral of this form can be written in terms of another. Whenever you use such a formula to make an integration problem easier, you call it a “reduction formula”. Thus,

$$\begin{aligned}
\frac{d}{dx} \sin^a x \cos^b x = \\
&= a \sin^{a-1} x \cos^{b+1} x - b \sin^{a+1} x \cos^{b-1} x \\
&= a \sin^{a-1} x \cos^{b-1} x - (b + a) \sin^{a+1} x \cos^{b-1} x \\
&= (a + b) \sin^{a-1} x \cos^{b+1} x - b \sin^{a-1} x \cos^{b-1} x
\end{aligned}$$

From any point (m, n), representing an integral in (E), these formulas allow you to relate the given integral to one whose exponents are given by a point two units away in the directions **NW**, **SE**, **N**, **S**, **E**, **W** — at least most of the time. You cannot leave in a particular direction if the coefficient of the term you want to change is zero. Careful consideration of cases shows that: (1) if $m = -1$, it cannot be increased; (2) if $n = -1$, it cannot be increased; (3) if $m+n = 0$, it cannot be increased. Since the reduction formula moves two units when it moves, two consecutive values on each line can be taken as the target of reduction. A complete set of target values (m, n) is $(-1, 0)$, $(-1, 1)$, $(0, -1)$, $(1, -1)$ and $(0, 0)$, corresponding to the integrals of $\csc x$, $\cot x$, $\sec x$, $\tan x$, and 1, respectively. In each of these cases, a special formula for the integral is known.

Another use of trigonometric identities. The formulas

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\cos(A-B) &= \cos A \cos B + \sin A \sin B
\end{aligned}$$

can be combined by adding or subtracting the first pair or the last pair to obtain an expression for a product of two trig functions as a sum. That is very useful in calculus, since sums appear in easy formulas and products in complicated ones.

Examples 7.2. For my first trick, I will apply this to

the notorious

$$\int \sec^3 x \, dx$$

that is discussed in Example 9 (page 427). Then, from the exercises, we compare reduction formulas and double angle formulas for

$$\int \cos^4 x \, dx \qquad \#3$$

A case in which the exponents are not integers, but the integral can still be found by these methods is

$$\int \sin^3 x \sqrt{\cos x} \, dx \qquad \#13$$

A polynomial is a sum of terms. Usually we use single terms as examples, since adding them together just joins two separate problems. Here is such an example.

$$\int \frac{1 - \sin x}{\cos x} \, dx \qquad \#17$$