Exam announcement. The original intent was to complete lectures through Section 7.6 in time for all of this material to be examined in a recitation class. We didn't quite get that far. However, this will not limit the scope of the exam on September 28. Lecture time was divided 40% for applications and 60% for techniques of integration, and the exam will have a similar ratio — though probably giving a little more weight to the techniques. No attempt will be made to avoid the improper integrals to be discussed later in this lecture, but their role in calculus is only to get around an overly severe limitation in the definition of the integral to give meaning to a calculation that feels correct.

A limited set of formulas will be provided with the exam. It is currently expected that few enough formulas will be included that they will not require a separate "formula sheet". Primarily, the formulas will concentrate on trigonometric identities. Formulas for derivatives or integrals of particular functions will *not* be included, nor will there be any version of integration by parts. If you fear that you will forget such

a formula that you have prepared, you should write it on the exam paper (or printed formula sheet if one is used) as soon as you get one. The only formulas of calculus that might be included are reductions formulas for some integrals of trigonometric functions. Your calculator also knows many basic results, and you should know how to interpret what it tells you. Unlimited use of calculators is allowed. However, unless instructed otherwise exact answers are expected. Thus:

$$\cos(\pi/6) = \sqrt{3}\ /\ 2$$
$$\int_0^1 \frac{dx}{1+x} = \ln 2$$
$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$$

In particular this means that a numerical integration routine (which, as we shall see, only approximates integrals) included in the calculator should not be trusted to evaluate a definite integral on the exam, although it may be useful to check that your answer is plausible.

Strategy for integration. It is easy to write integrals that cannot be evaluated as known functions. While a proof of this is "beyond the scope of this course" (an annoying phrase that was common in the honest textbooks I used as a student, although I haven't seen it in current textbooks), it suggests that a mechanical procedure can be found that tests whether an integral can be expressed in terms of familiar function. Since it is hard to imagine such a process that wouldn't be able to tell you why these functions suffice to express the integral, this test is likely to extend to an algorithm to integrate all *possible* integrals. The full details of this are complicated, but it means that there is almost always an obvious clue that allows all integrals in a calculus textbook to be found.

Some integrals have been made mechanical. We have already seen two families of integrals for which we have given (most of) a proof that all functions in the family can be integrated. These proofs included a guide to a step-by-step process for finding the integral. These families are: (1) functions that are sums of terms of the form $\sin^m x \cos^n x$; (2) rational functions.

What else can we do? The remaining tools are: (1) substitution; (2) integration by parts. These are the translations to the integral calculus of: (1) the chain rule; (2) the product rule.

Substitution. Two types of substitution have been met: (1) when the integrand appears to be expressed in terms of an expression $g(x)$ except for a factor of $g'(x)$; (2) integrals containing the square root of a quadratic expression $ax^2 + bx + c$. The first type includes

$$\int xe^{x^2}\,dx = (1/2)e^{x^2} + C,$$

but not

$$\int e^{x^2}\,dx,$$

which does not have an elementary integral.

Another example of this type of substitution has an integrand that is a rational function of an exponential. The few exercises in the textbook of this type seem to miss the mark, so consider

$$E = \int \frac{dx}{1+e^{7x}}.$$

I claim that the integrand can be written as a function of $y = e^{7x}$ multiplied by $dy = 7e^{7x}\,dx$. Although you don't see the factor of $7e^{7x}$ that would allow you to isolate the dy in this integral, you can simply introduce it by multiplying numerator and denominator by this quantity without destroying the property that *everything else* can be written as a function of y. Thus,

$$\begin{aligned} E &= \int \frac{7e^{7x}\,dx}{7e^{7x}\big(1+e^{7x}\big)} \\ &= \int \frac{dy}{7y(1+y)} \\ &= \frac{1}{7}\int \frac{1}{y} - \frac{1}{1+y}\,dy \\ &= \frac{1}{7}\big(\ln y - \ln(1+y)\big) + C \\ &= \frac{1}{7}\ln\left(\frac{y}{1+y}\right) + C \end{aligned}$$

$$= \frac{1}{7}\ln\left(\frac{e^x}{1+e^x}\right) + C$$

Trigonometric substitution. The second type of substitution includes

$$I = \int \frac{dx}{\sqrt{9x^2+12x-5}}, \qquad \#29$$

in which *completing the square* gives

$$9x^2+12x-5 = (3x+2)^2 - 9,$$

suggesting

$$\begin{aligned} 3x+2 &= 3\sec u, \\ 3\,dx &= 3\sec u\tan u\,du, \\ \sqrt{(3x+2)^2-9} &= 3\tan u. \end{aligned}$$

Making these substitutions gives

$$\begin{aligned} I &= \int \frac{\sec u\tan u\,du}{3\tan u} \\ &= (1/3)\int \sec u\,du \\ &= (1/3)\ln|\sec u + \tan u| + C \\ &= (1/3)\ln\left|x + 2/3 + 1/3\sqrt{9x^2+12x-5}\right| + C \end{aligned}$$

This may be the best example of an integral for which the answer provides a poor hint about the method of integration.

Don't forget algebra. Examples in the textbook show how the identities

$$\begin{aligned} \frac{1}{1-\cos x} &= \frac{1+\cos x}{\sin^2 x} \\ \sqrt{\frac{1-x}{1+x}} &= \frac{1-x}{\sqrt{1-x^2}} \end{aligned}$$

can put an integral in a form that reveals a trick that was previously hidden. The trigonometric identities

that replace products of trigonometric functions by sums of functions at different values are used so frequently that they are part of the standard method for some integrals.

Improper integrals. Consider the definite integral

$$P(N) = \int_0^N \frac{dx}{1+x^2},$$

where N is a large (positive) real number, i.e. something off to the right of whatever part of the number line you carry around with you. Since you know the corresponding indefinite integral,

$$P(N) = \arctan N - \arctan 0 = \arctan N.$$

The graph of the arctangent function has a horizontal asymptote at $\pi/2$ (corresponding to the vertical asymptote in the graph of the tangent function) that is approached for large positive values, we know that $P(N) \to \pi/2$ as $N \to \infty$. Since the function approaches this limit so *nicely*, there seems no harm in saying that

$$P(\infty) = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Similarly, although the integrand in

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

is unbounded on the domain of integration, causing our definition of the Riemann integral to break down, the indefinite integral is $\arcsin x$, so we again expect an answer of $\pi/2$. Again, the proper Riemann integral from 0 to a value slightly less that 1 has a value that is only slightly less than $\pi/2$ and approaches $\pi/2$ as the right endpoint approaches 1.

An additional limit explains it all. If the only obstruction to interpreting an expression as a Riemann integral is due to one infinite endpoint or an integrand that becomes infinite at one end of the domain of integration, consider the integral on a domain that stays

away from that endpoint. This integral does exist and defines a function of the awkward endpoint (whether we know an expression for that function or not). The question of whether this function can be extended in a reasonable (the technical term is *continuous*) way to include the point we want is the question of whether a limit exists. Some of the limits arising in this way can be handled by the methods of Chapter 1 — the two examples we gave fall in this class — but others lead us into more theoretical considerations based on the behavior of the integrand. The comparison tests to determine if the limit exists (the word *converges* is frequently used) or not (signified by the expression *divergent improper integral*) are of this nature. We will say more about this when similar considerations are needed in the context of infinite series.

In most of the examples of this type, one can evaluate the integral as if nothing were wrong. If the improper integral converges, the formulas you use involve continuous functions, so they automatically evaluate limits when evaluated at points that can be approached by points allowing the Riemann integral to be defined. If

you get a disturbing answer, as in

$$\int_0^1 (1-x)^{-3/2}\,dx,$$

you should recognize that you were careless in working with an improper integral.

If both endpoints are troublesome, simply choose a convenient point P inside the domain of integration and consider the total integral as the integral from the left endpoint to P plus the integral from P to the right endpoint.

Hidden impropriety. If you apply the familiar formulas of calculus to integrating x^{-1} or x^{-2} from -1 to 1, you get a disturbing answer. This is explained by the behavior of the integrand at $x = 0$, which is a point inside the domain of integration. The only sensible definition of the integral in such a case is as a sum of the improper integral from -1 to 0 and the improper integral from 0 to 1. In both cases, both of these improper integrals diverge. Sound mathematical practice demands that an integral build from

divergent improper integrals be considered to be divergent.

If you are in the practice of sketching a graph of the the integrand over the domain of integration, you will see a clear warning of this type of improper integral. Almost the same effect can be achieved by checking that every function can be evaluated wherever it is needed. In the case of functions appearing as integrands, this means that the function should be defined throughout the domain of integration. Existence of the Riemann integral of f is proved when f is continuous throughout the domain of integration. In theory, this is much more demanding than mere existence, but the method of verification is similar. Indeed, almost all functions in our toolbox are continuous, so the verification that the function can be evaluated often includes a proof of continuity — whether you realize it or not.

Moreover, if you can differentiate the function, your calculation includes a proof that the function has a derivative. Since this is a stronger property than continuity, you have also shown the the function is continuous.