**Comments on exam**. The quoted reduction formula should have been

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

Since it was not needed in any problem, no questions were raised during the exam.

The problems with lowest average grades were

#7 (41%), #4 (54%), and #6 (58%).

The topics represented by these problems may need more time. The trigonometric substitution that simplifies

$$\int \frac{dx}{x^2\sqrt{x^2-16}} \qquad \#7$$

is

$$\begin{aligned} x &= 4\sec u \\ dx &= 4\sec u \tan u\,du \\ \sqrt{x^2-16} &= 4\tan u \end{aligned}$$



with $\tan u > 0$. If we were to use this substitution on a definite integral, positive $x$ would correspond to $u$ with $0 \le u < \pi/2$ and negative $x$ would correspond to $u$ with $\pi \le u < 3\pi/2$. Since the integrand is only defined for $|x| \ge 4$, the interval of integration must contain numbers of only one sign, so separate intervals for different signs causes no trouble. Improper integrals with an infinite endpoint for $x$ will be transformed into integrals with an endpoint of $u = \pi/2$ if $x > 0$ of $u = 3\pi/2$ if $x < 0$. The resulting integral may still be improper if the integrand is unbounded at this endpoint. Although this is the least common trig substitution, an integral of this form was *hand-in* homework problem 7.3#12, so should have been familiar. This problem will be completed at the blackboard.

There are some general comments to be made about volume problems before jumping into the particular example in #4. First, the coordinate axes should be considered to be transparent. They are not to be used as part of the boundary of a region unless you are told to do so. Questions about this were raised during the exam and the shape of the region $\mathcal{T}$ was clarified on the blackboard. Second, to set up the integral, you only need to have a picture of $\mathcal{T}$, perhaps enhanced with horizontal or vertical line segments inside the region — three dimensional shading to illustrate the effect of rotating it adds nothing to the solution. To get the right answer, you usually need to identify the region being rotated. Third, rotating about different axes will usually give different answers, so you need to use a formula appropriate to the given problem. Rather than trying to remember which of the four volume formulas we have applies in each case, it is better to describe the formulas in terms of the geometrically meaningful parameters $r$, the distance *from* the axis of rotation, and $h$, the distance *along* the axis of rotation. The formulas should be remembered in the simplified form together with rules for extending to



more complicated regions. This approach will be the basis for the way these problems are treated by methods of multivariable calculus. The volume formulas are

$$\pi \int r^2 \, dh \text{ (disks)}$$

$$2\pi \int r h \, dr \text{ (shells)}$$

Application to one version of this problem will be shown at the blackboard.

Problem #6 asks to integrate a rational function, so it can only be an exercise in partial fractions (although one version could have been done by a substitution if you thought to look for it). The usual way to obtain the partial fractions is a memorable trick, so it should be mastered by the next time you meet a problem like this. In addition, the underlying principle leading to the partial fraction decomposition should be easy to remember, and leads to equations that are easily solved by methods other than the usual trick. Another approach that seems natural was (almost) discovered on one of the exam papers. Here is how it looks when applied to one version of #6. It is easily seen that

$$\frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+1)(x+2)}$$

The given integrand was $4x+7$ times the quantity on the right, so we form, and rewrite, the product of this with the terms on the left.

$$\frac{4x+7}{x+1} = 4 + \frac{3}{x+1}$$
$$\frac{4x+7}{x+2} = 4 - \frac{1}{x+2}$$

Subtracting, and noticing that both expressions have the same integer part,

$$\frac{4x+7}{(x+1)(x+2)} = \frac{3}{x+1} + \frac{1}{x+2}.$$

The integration will be done at the blackboard.

One mistake was far too common on the exam, and could appear in almost any problem. It might be paraphrased as, "I know that $g(x)$ is either the derivative



or the integral of $f(x)$, but I'm not sure which". This confusion should never arise: there is a clear way to start with an expression for one function and find an expression for its derivative, but integration almost always requires guessing. It is a slight exaggeration to say that *there are no integration formulas*, but the intended interpretation is that *all formulas should be remembered as differentiation formulas*. Our development of the differential calculus started from a few basic formulas for the derivative of constant functions, the identity function $x$, $e^x$, and $\sin x$, and derived everything else using the rules for sums, products, compositions, and inverse functions that are the same tools used to work though exercises and exam problems.

Integration techniques mostly consist of looking for clues that show how a given result could arise from these rules and algebraic simplification.

**Definition of limit of sequence**. Although a sequence is nothing but a function defined on the positive integers, it is customary to write its values as $a_n$ instead of $a(n)$. Many sequences have values $a_n$ that are computed as familiar functions, like $(n-1)/n$, $1/n^2$,

$(-1)^n\sqrt{n}$. Others are defined by a rule that computes each term of the series in terms of previous terms of the same series. A famous example is the Fibonacci sequence which, after being given $f_1 = f_2 = 1$, defines remaining terms by $f_n = f_{n-1} + f_{n-2}$. The central question that dominates our work with sequences is: "do the terms approach a single value?" This question is not sensitive to the values at the beginning of the sequence. Any finite number of $a_n$ can be changed without affecting this property. If there is such a value, it is called the limit of the sequence. A formal definition is very similar to the definition of the limit of a function. Although we will speak in terms of finding limits when they exist, what is defined is the entire sentence

$$\lim_{n\to\infty} a_n = L.$$

Note that it is always the limit as $n \to \infty$ that is of interest here. The definition has the expected form: for all positive numbers $\epsilon$, there is a positive integer $N$ such that

$$\text{if } n > N \text{ then } |a_n - L| < \epsilon.$$



**Uniqueness of limits**. The result that justifies talking about limits as definite quantities that can be computed from the sequence is something whose formal expression is:

$$\text{if } \lim_{n\to\infty} a_n = L_0,$$
$$\text{and } \lim_{n\to\infty} a_n = L_1,$$
$$\text{then } L_0 = L_1$$

which says that a sequence cannot have more than one limit. You may protest that the sequence $a_n = (-1)^n$, being alternately equal to $+1$ or $-1$ is approaching these two values, but the given definition of limit says that this sequence has no limit because whatever value you try for $L$, there will always be arbitrarily large $n$ with $a_n$ *not* close to $L$, indeed, some terms will always have $|a_n - L| \geq 1$.

**Real numbers as decimals**. The decimal representation of numbers, which for numbers between 0 and 1 has the form $0.a_1a_2a_3\ldots$ where all of the $a_n$ are taken from the set of *digits*, $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, is a familiar way to describe real numbers. The first $N$

elements in this string of digits represents the rational number

$$S_N = \sum_{n=1}^{N} \frac{a_n}{10^n},$$

and the whole infinite string represents

$$\lim_{N\to\infty} S_N.$$

Since $S_N$ increases when $N$ increases, and since $S_N + 10^{-N}$ is an upper bound on all $S_n$, in particular all those with $n > N$, the $S_n$ *seem to* approach a definite value. The current idea of real numbers is that all such sequences should be allowed to define numbers, even if we cannot predict new digits before they are revealed to us. This is mildly disturbing, but all alternatives seem much more disturbing.

**Theorems and examples**. A more general wording of our belief that every infinite string of decimal digits defines a real number is, "every bounded increasing sequence has a limit". A useful example is

$$a_n = \frac{n-1}{n}.$$



Recalling our earlier discussion of proper fractions, this should be written as $a_n = 1 - 1/n$. As $n$ increases, $1/n$ decreases to zero, causing $1 - 1/n$ to increase with limit 1. This method is more generally useful. It does not depend on sequences being increasing or decreasing, since it can be shown that every sequence for which $a_n$ is a *proper* rational function of $n$ has limit zero.

The theorems about limits of functions have corresponding theorems for sequences. There are no surprises. If we need to say more later, we will, but it is better to see these ideas used than to spend forever getting ready to use them.

These results are used in the exercises of section 10.1. Several will be done at the blackboard. Material from later sections will be introduced if time permits.