Additional properties of sequences. In addition to the algebraic properties of limits, if

$$\lim_{n\to\infty} a_n = L,$$

then

$$\lim_{n\to\infty} a_{n+1} = L.$$

This is most useful when one has some sort of recursive definition of a_n. This appears in Example 11, Exercise 50 and Exercise 60 of Section 10.1.

In general, if $a_{n+1} = f(a_n)$, this says that any limit L must satisfy $L = f(L)$. For example, while there is no *closed form* expression for the solution of $x = \cos x$, it can be found as the limit of a sequence with $a_{n+1} = \cos(a_n)$. When f takes an interval I to itself and *shrinks all distances*, in the sense that

$$|f(x) - f(y)| \le c\,|x - y| \qquad (B)$$

for some $c < 1$, then the equation $f(x) = x$ will have a unique solution in I and all sequences formed by taking $a_0 \in I$ and iterating f will converge to that

solution (and converge fairly rapidly). In particular, if $-c < f'(x) < c$ for $x \in I$, the Mean Value Theorem give (B). In fact, if $\left|f'(L)\right| < 1$, iterations that start close enough to L will converge to L. The smaller c is, the faster the sequence converges to L. Newton's method converges extremely rapidly because it was constructed to have $f'(L) = 0$ even though we didn't know L.

Although sequences can arise in many ways, the sequences arising from iteration seem to be what we expect. In particular, those with positive derivative at the fixed point give sequences that are monotone; that is, either always increasing or always decreasing; and those with negative derivative oscillate around the limit, alternately larger and smaller, but getting closer at each step. Sequences having these two types of behavior are easier to prove things about, so will be more likely to be selected as examples. Theorem 10 of Section 10.1 (p. 585) asserts that all that is needed to be sure that an increasing sequence converges is something to keep it from getting large. This will play an important role when we move from sequences that can be shown to converge by producing their limits to sequences that can be used to *define* their limits by being shown to converge without knowing the limit in advance.

Introduction to series. Given one sequence a_n, defined for $n > 0$, one can form another sequence s_n by trying to add up all the a_n. The precise definition is $s_0 = 0$ and

$$s_n = s_{n-1} + a_n$$

for $n > 0$

A familiar infinite series is the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1}.$$

The use of $n-1$ in the exponent is a technical device that allows the first term, with $n = 1$, to be a.

It is easy to prove by mathematical induction that

$$s_n = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad (G)$$

An informal argument, that isn't really a proof, is familiar. With practice, the induction argument is just as easy as pretending that you can see a bunch of terms that haven't been written, and induction meets current standards of mathematical rigor, so it should be preferred.

If $|r| < 1$, $r^n \to 0$, so (G) shows that the limit of the s_n is $a/(1-r)$.

Periodic decimals are a common example of geometric series. The expression for the limit shows that the number represented by a periodic decimal is always rational. Other familiar numbers, like $\sqrt{2}$, e or π are known not to be rational, so their decimal expansions are not periodic. If a number x has a purely periodic decimal expansion with a period of length k, then $10^k x$ is an integer plus exactly the same decimal, so $(10^k - 1)x$ is an integer. Such numbers thus represent fractions with denominator $10^k - 1$. They may not be in lowest terms, so the true denominator may be a divisor of this. For example, exercise 39 of section 10.2 asks about $0.\overline{307}$

The terms of a convergent series approach zero. If $\lim_{n\to\infty} s_n = L$, then also $\lim_{n\to\infty} s_{n-1} = L$, and general properties of limits say that the difference approaches zero. The difference is just a_n, so terms approach zero.

Some divergent series also have terms that approach zero. Although $a_n = 1/n$ approaches zero, the series with these terms does not converge. The terms decrease so slowly that the sum of all terms with $2^k \le n < 2^{k+1}$ can be shown greater that $1/2$ for each k. This means that it is possible to recognize some divergent series just by noting that the terms do not approach zero; but it takes more to show that a series is convergent.

Series with positive terms. If all $a_n > 0$, then $s_n > s_{n-1}$, so that the partial sums form an increasing sequence. These series either converge or have arbitrarily large partial sums. Initially, apart from isolated examples, we consider only these series.

The integral test. If the a_n are also decreasing, and given by evaluating a simple expression $f(x)$, that is also decreasing, at integer n, then the s_n of the series grow at the same rate as $\int_0^N f(x)\,dx$ (the lower limit of 0 is chosen arbitrarily, if the integral is not defined there, you should chose a larger value for the lower endpoint, since it is only the behavior for large N that is relevant).

Comparison tests. If $0 \le a_n \le b_n$, and $\sum b_n$ converges. the sum gives a bound on $\sum a_n$, so $\sum a_n$ converges. In the same situation, if $\sum a_n$ is known to diverge, so must $\sum b_n$. Although this is very useful for many examples discussed in this course, it is nothing but an interpretation in the language of series of the fact that bounded increasing sequences converge.

Main uses of this test are: (1) if $a_n > c/n$ for some positive constant c, then $\sum a_n$ diverges; (2) if $a_n = 1/f(n)$ where f is a polynomial of degree greater than 1, then $\sum a_n$ converges; (3) if a series of positive terms converges, then the series formed by selecting some of its terms also converges; (4) the ratio test.

Observation (3) is used in the discussion of *absolute convergence*, which will appear in the next lecture. The ratio test is discussed in the next section.

The ratio test. We state this test only in the case of $a_n > 0$, and leave the extension to more general series for a later time (not much later).

Suppose that it is possible to evaluate

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Then, $\sum a_n$ converges if $r < 1$ and diverges if $r > 1$. The case $r = 1$, along with cases in which r does not exists, are inconclusive. The test works by comparing the given series to geometric series of ratio r.

Although this is a fairly weak test, this *ratio test* will be very useful when applied to power series.

The root test. A related test uses $\sqrt[n]{a_n}$. Although slightly more widely applicable than the ratio test, this *root test* is often much more difficult to apply. Although I will feel free to use the root test in lecture, I don't think that it is reasonable to include a problem on an exam that requires you to use this test to recognize the convergence of a series, and I expect that all of my colleagues share this view.

Both tests introduce artificial difficulties since they require certain limits to exist. This obscures the main idea of the test, which is that some properties of the a_n can be used to generate a geometric series to which $\sum a_n$ can be compared. Indeed, in the case of divergence, all that has been accomplished is that it has been discovered that the a_n do not approach zero.