Absolute convergence. If $\sum |a_n|$ is convergent, the series $\sum a_n$ is said to be **absolutely convergent**. Although defined as the convergence of the series of absolute values, the name suggests a strong type of convergence. This is indeed the case: an absolutely convergent series is convergent. The proof is a neat little trick: note that

$$\frac{|a| + a}{2} = \begin{cases} a & \text{if } a \geq 0 \\ 0 & \text{if } a \leq 0 \end{cases}.$$

That is, this expression gives a way of selecting the positive terms of the original series. With that interpretation, the comparison test shows that the series with these terms converges if $\sum |a_n|$ converges. Similarly, $(|a| - a)/2$ is a series of positive terms, which is $|a_n|$ if $a_n < 0$, and the comparison test shows that it converges if $\sum |a_n|$ converges. The original series $\sum a_n$ is the difference of these two series, so standard properties of limits show that it converges to the difference of the sums of the two series just constructed.

Any test showing convergence of series of positive terms, when applied to the absolute values of the terms

of $\sum a_n$ proves the absolute convergence, and hence also the convergence of the series.

Absolute convergence and rearrangements. If a series is absolutely convergent, any way of adding together all the terms of the series will give the same answer. If a series is convergent, but not absolutely convergent, the convergence depends on the particular sequence of partial sums used to find the limit. If terms are accumulated at a different rate, the limit can be different. There is an example on p. 619 that shows that a rearrangement can make a 50% difference in the value of the sum. This example has the advantage of being explicit, but it has a mild conclusion. Given any number L that you want to reach as the sum, if you alternately accumulate positive numbers until the sum if bigger than L, and then negative numbers until the sum is less than L, you wind up with a sequence of partial sums converging to L.

(Infinite) Series and (Improper) Integrals. Another way to get a series is to let

$$a_k = \int_{k-1}^{k} f(x)\, dx$$

leading to

$$s_n = \int_0^n f(x)\, dx.$$

The type of limit used in defining the sum of a series is (almost) the same as that used in defining improper integrals. (In defining the integral, all large values of the upper endpoint, not only integers, must give values close to the limit. If $f(x) > 0$, which is the main case of interest because of the role of absolute convergence, the integral is an increasing function of its upper endpoint, so sampling at integers allows you to distinguish between the bounded and unbounded cases.)

The special case in which $f(x)$ is a decreasing function arises often enough to get special attention. In this case,

$$f(k-1) \geq \int_{k-1}^{k} f(x)\, dx \geq f(k),$$

so the integral can be compared to the series whose terms are the values of $f(x)$ at integers. The main

example of this is

$$\sum_{n=1}^{\infty} n^{-p} \text{ and } \int_1^{\infty} x^{-p}\, dx.$$

The sum is awkward since these appear to be the simplest examples of series whose terms approach zero at this rate (they aren't, really, at least for integer values of p, but that's a long story), so we want to know how they behave in order to compare other series to these. Comparison to the improper integral does what we need, since

$$\int_1^N x^{-p}\, dx = \frac{1}{1-p}\Big(N^{1-p} - 1\Big).$$

From this, we see that the integral, and hence also the series, converges for $p > 1$ and diverges for $p < 1$. The special case of $p = 1$ leads to a different formula for the integral that also shows divergence.

Alternating series. Series whose terms are strictly alternating in sign — usually indicated by having terms

$a_n = (-1)^n f(n)$, where $f(n)$ is some function easily seen to have the same sign for all positive integers n — are especially nice. If the absolute values of the terms decrease and approach zero, an alternating series must converge. For example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is convergent, although it is not absolutely convergent. In this case, we could show the series convergent by grouping the terms in pairs and using

$$\frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{2n(2n-1)}.$$

The series with these terms converges by comparison to a multiple of $\sum n^{-2}$, and the partial sums of an odd number of terms of the original series differ from these partial sums of an even number of terms by a single term, and the terms of the series go to zero. This seems fairly easy, but the general result is no more difficult, and since it avoids irrelevant computation, may even

be easier. Since the terms are decreasing in magnitude and alternating in sign, the sum of $n+2$ terms is always between the sum of n terms and the sum of $n+1$ terms. Induction on this shows that all subsequent partial sums lie between any two consecutive partial sums. If you stop after a negative term, you get a partial sum that is smaller than all subsequent partial sums, and the sequence of these quantities is increasing and bounded (by any of the other partial sums). Its limit is the sum of the series, and the other partial sums give a sequence that decreases to this value. The N in the definition of limit need only be chosen so that all terms from a_N on are less than ϵ in absolute value.

Using all the tests. Begin by looking at the terms a_n. If the series converges, they must have limit zero. If *anything else* happens, the series cannot be convergent. A typical exam question asks you to “state the convergence test and show the method used”. This test is known as the **test for divergence**. Using it calls for evaluating the limit of the a_n by the usual methods we have applied to sequences and getting an answer other than zero, or showing that the sequence doesn’t

settle down. An extreme example is a sequence like $\cos n$ can be shown by a difficult argument to take values that are infinitely often close to any number between -1 and $+1$. You would not be expected to give a proof of this, but you should recognize, say by looking at a few values, that it must behave something like this. Our goal is give you enough experience with different types of series that you can recognize which ones converge. You are given the words to describe a few of the tests so we can be sure that your responses to the few examples on the exam represent an ability to deal with other similar series.

If the series is absolutely convergent, you will be expected to recognize that, so you should then test for this by looking at the absolute values of the terms. The integral test requires that the $|a_n|$ be steadily decreasing, and the convergence part of the ration test proves it. If the series doesn’t have this property, you should be very suspicious. It could be that you have modified a series, already known to be convergent, by multiplying by a bounded oscillating quantity. Comparison to the known convergence series shows that this will

also converge. If you modify a divergent series in this fashion, the result is unpredictable. If you meet something that looks like this, you should suspect that a special trick is involved. We have seen examples of special tests for such series based on summing blocks of terms, but this goes beyond what is likely to be met on an exam.

If the terms of a series are rational functions of n, only the leading terms in the numerator and denominator are significant in determining the size of the terms as a function of n. All such series are then essentially p-series, and we have a complete characterization of these series.

Alternating series are a special class of series with their own rules. The convergence theorem in this case requires that the absolute values of the terms decrease steadily to zero.