Exercises to be done at the blackboard. We begin with some illustrations of material shown in the previous lecture. First, from Section 10.5 on alternating series. The instructions are to test for convergence.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{5n+1} \qquad \#7$$

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1} \qquad \#9$$

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+4} \qquad \#11$$

$$\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^{3/4}} \qquad \#15$$

$$\sum_{n=1}^{\infty}(-1)^{n}\sin\left(\frac{\pi}{n}\right) \qquad \#17$$

Then, there are some from section 10.7 with the same instructions.

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1} \qquad \#1$$

$$\sum_{n=1}^{\infty}\frac{4^n}{3^{2n}-1} \qquad \#3$$

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^2} \qquad \#5$$

$$\sum_{n=2}^{\infty}\frac{n^3+1}{n^4-1} \qquad \#11$$

$$\sum_{n=1}^{\infty}\frac{3^n}{5^n+n} \qquad \#17$$

Power Series. If a series $\sum a_n$ is absolutely convergent, than any series obtained by multiplying each of these terms by something less than 1 will also be absolutely convergent. In particular, all of the series $\sum a_n x^n$ with $|x| < 1$ are convergent. This can be strengthened to show that if the a_n are bounded, the $\sum a_n x^n$ with $|x| < 1$ are absolutely convergent. In particular, if $\sum a_n$ converges, the a_n approach zero, so must be bounded. Thus convergence for one value of x gives absolute convergence for smaller x.

It follows from this that such a *power series* always converges in an interval around zero that is symmetric around zero except possibly for containing one endpoint of the interval and not the other.

If each such series has a sum, that sum defines a function of x. The only limitation of this approach is that there is no interpretation of this function outside the interval where the series converges. For example, the geometric series leads to the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and the series converges absolutely when $|x| < 1$. Although the function on the left side of this equation is defined for all $x \neq 1$, the series can only recover those values on a small interval.

More general power series. It is often convenient to look at series of the form

$$\sum_{n=0}^{\infty} a_n (u-c)^n$$

obtained by putting $x = u - c$ in the previous formula. This series will converge for those u for which the x computed as $u - c$ causes the previous series to converge. This series will converge in a symmetric interval in u around c. For example, $|x| < 1$ corresponds to the interval $c - 1 < u < c + 1$. Since a series is only a description of the limit of the sequence of partial sums, no attempt should be made to simplify the individual terms of the series — they are to be evaluated first, and then combined. Some operations on the terms can be justified, but it is best to take a cautious approach to manipulating the terms of a series.

Relation to the ratio test. The ratio of two consecutive terms of a power series is

$$\frac{a_{n+1}x^{n+1}}{a_n x^n} = \frac{a_{n+1}}{a_n}x.$$

If

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists, the power series converges whenever $L\,|x| < 1$ by the ratio test. This gives $1/L$ as a radius of convergence. Note that the use of the ratio test gives such a simple criterion for convergence of the series, that the test should be used rather than attempting to use it to obtain a formula for the radius of convergence.

Indeed, when this limit exists, the ratio test is inconclusive only when $|Lx| = 1$, that is, at the endpoints of the interval of convergence. In general, behavior at the endpoints is special, and each individual case must be studied separately.

The information produced by the root test is similar. These tests seem to have been designed mainly for application to power series.

Power series and calculus. An important result is that any equation between a function and a power series that has a positive radius of convergence can be differentiated or integrated within that interval of convergence. The proof is not easy, but the result is *true* and that is all that we need at this point. Indeed, the series obtained in this way will also be valid at endpoints of the interval of convergence, when they can be interpreted there. Starting from the geometric series, and some of its variants, we can find series representations for other functions. Examples given in the text are:

$$(1-x)^{-2} = \frac{d}{dx}(1-x)^{-1} \tag{5}$$

$$\ln(1-x) = \int_0^x (1-t)^{-1}\,dt \tag{6}$$

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt \tag{7}$$

The series (6) at $x = -1$ shows that an alternating series met in previous examples converges to $\ln 2$, and (7) at $x = 1$ gives a similar alternating series that converges to $\pi/4$.