Some examples of power series. Here are some exercises from Section 10.9. General instructions are to find a power series and determine its interval of convergence. In #9 you are told to use partial fractions.

$$\frac{1}{1+4x^2} \qquad \#3$$

$$\frac{x}{x-3} \qquad \#7$$

$$\frac{3x-2}{2x^2-3x+1} \qquad \#9$$

$$\ln\left(\frac{1+x}{1-x}\right) \qquad \#17$$

$$\int \frac{dx}{1+x^4} \qquad \#21$$

$$\int \frac{\arctan x}{x}\,dx \qquad \#23$$

These examples show that you now have no reason to speculate about derivatives or integrals. Once you have a series representation of a function, term-by-term differentiation or integration will give a series

for the derivative or integral, and any correct answer will have the same series.

Taylor series. For clarity of exposition, we will develop the results only for series in powers of x. The extension obtained by writing $x = u - c$ for some constant c leads to formulas that appear more general, and it is this generalization that is usually called a Taylor series. The special case of series in powers of x (that is, Taylor series with $c = 0$) is called a Maclaurin series. The distinction is minor, and the general formulas follow easily if you have the special case for *all* (sufficiently well differentiable) functions.

First, note that we are assuming that the equation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (T)$$

is interpreted to mean that, within the interval of convergence of the series, the series converges to $f(x)$ for all individual values of x. In particular, we can take $x = 0$. This gives $f(0) = a_0$.

Furthermore, the claims that we made about differentiation of series within the interval of convergence lead to

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$
$$f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$
$$f'''(x) = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

etc. We have preserved the sum as starting with $n = 0$, but the initial coefficients contain a factor that is zero, so only terms with nonnegative powers of x will survive. Evaluating these expressions at zero leads to $f'(0) = a_1$, $f''(0) = 2a_2$, $f'''(0) = 6a_3$, and so on. The general formula is $f^{(n)}(0) = (n!)a_n$. If $f(x) = e^x$, $f^{(n)}(x) = e^x$ for all n, so $f^{(n)}(0) = 1$ for all n, and $a_n = 1/n!$. This gives a series that converges for all x. A similar method can be used to find the series corresponding to $\sin x$ or $\cos x$. Once these series are

available, substituting a constant multiple of a power of x for x gives new series representations.

Taylor series converge. Everything we have said about series is based on an assumption that the series converges on some interval. To prove convergence, an estimate on the difference between $f(x)$ and the sum of the first several terms of its Taylor series is necessary. For the first term — the constant term — this is known from the Mean Value Theorem

$$f(x) - f(0) = (x - 0) f'(\xi)$$

for some ξ between 0 and x. Note that this difference looks like the linear term of the Taylor series except that it is evaluated at the unknown point ξ instead of at 0. If $f'(x)$ is continuous, it will be bounded on any finite interval around 0, and this allows a bound to be found on $|f(x) - f(0)|$ of the form $C\,|x|$. This bound approaches zero as $x \to 0$. The Lagrange form of the Taylor series error is a formula of the same type that is a generalization of the Mean Value Theorem:

$$f(x) - \sum_{k=0}^{n-1} a_k x^k = \frac{f^{(n)}(\xi)}{n!} x^n$$

with ξ between 0 and x.

For many functions, this formula can be used for both the practical purpose of allowing good approximations to be computed (using only the familiar operations of addition and multiplication) and the theoretical purpose of justifying term-by-term operations on the series for the function.

Operations on series. In addition to term-by-term differentiation, one can add and multiply series. Addition involves adding the coefficients of x^n in two series, and is justified by theorems on limits of sums. Multiplication is a little more interesting. If you multiply the terms through the one containing x^n in two series, you get a polynomial of degree $2n$. To get the desired formula, the terms of degree greater than n in this product must not be taken seriously. If additional terms are taken of the series we are multiplying, these terms will change, but the terms of degree at most n in the product depend only on the terms of degree at most n in the factors. (This holds only because we insist that only non-negative exponents appear in our series.) This means that there will be some technical

details in a proof that these operations give correct answers, The formulas are true even if the method of proving them goes beyond the way that we like to think about the results.

It is also possible to substitute an expression *without constant term* into a series and simplify the result to get a formula for the series associated with the composite function. Again, each term of the result depends on only finitely many terms of the series being composed.

Examples. Here are exercises from Section 10.10. In all cases, the Maclaurin series (i.e. Taylor series about 0) is to be found using series previously described.

$$x^2 \cos x \qquad \#19$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \#23$$

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \qquad \#25$$

$$\sqrt{1+x} \qquad \#27$$

$$\int \sin x^2 \, dx \qquad \#33$$

3 terms of $e^{-x^2}\cos x$ #41

Extending the binomial theorem. If k is a positive integer, $(1+x)^k$ can be expanded as a polynomial of degree k. For each j with $0 \le j \le k$, the coefficient of x^j is found by imagining all terms in the full expansion of the product of k factors, each $(1+x)$. These terms are all powers of x, and we count those that are exactly x^j. This count is the number of ways to form a set of k elements from a fixed set of n elements. This counting problem can be solved by studying the process for picking the elements of the set one at a time. If this is done, there are

$$k(k-1)\cdots(k-j+1)$$

ways of listing sets of j elements, and each set appears exactly $j!$ times in the list. This description of the binomial theorem leads to a formula that does not depend on k being an integer, although we don't get a polynomial unless k is a nonnegative integer. Newton observed that our formula for the coefficients of Taylor series gives these coefficients for all k. We shall

use this formula without repeating its derivation from the general formula.

General remarks. If we write $x = (u-c)$ in (T), we get

$$f(u-c) = \sum_{n=0}^{\infty} a_n (u-c)^n$$

with the same formulas for finding the a_n from the derivatives of $f(x)$ at $x = 0$. However, we are likely to be tempted to simplify $f(u-c)$ to get an expression $g(u)$ in which all connection with the function $f(x)$ has been lost. Even if this is done, we still have all we need to find the a_n. Since $x = u - c$, the chain rule factor dx/du is the constant 1, so repeated differentiation of $g(u)$ with respect to u is the same as finding the same derivative of $f(x)$ with respect to x and substituting $(u-c)$ for x in this expression. Thus, evaluating such an expression at $u = c$ is the same as evaluating the corresponding expression in x at $x = 0$. The general form of Taylor series has the same theory as the special case of Maclaurin series.