Exam grades. The average score was 59%, with a median of 65%. The highest scores were 93 and 89, with 5 more grade above 80. There is no clear division between passing and failing grades: certainly the 13 scores below 45 are failing, but not all of the 43 higher scores can be considered passing.

The problem with the poorest response was #8. Although the given formulas identified the first term of the series as being given by $n = 0$, this was frequently ignored. Even when some form of the original series were obtained, they were not multiplied correctly (only corresponding terms were multiplied, not the partial sums). Formulas are most useful as summaries of calculations that have been done often enough to be both familiar and tedious. A formula only serves to remind the user of its derivation. Since there is always the possibility of making a mistake in a calculation, every formula should be regularly checked against its interpretation. For example, we know that $\cos 0 = 1$, so the Taylor series of $\cos ax$ for any constant a must begin with the constant term 1.

The result on the problem of evaluating the sum of a

pair of geometric series (#2 on some exams, #3 on others) was the most disappointing. The problems asking for application of convergence tests had averages in the 65% range, but there were frequent mistakes.

The first problem did a good job of serving as a warm-up with an average of almost 87%.

Numerical integration. Suppose we want to evaluate the integral of some function $f(x)$ over the interval $[a, b]$. The result will be a number, and there are many cases in which it is better to try to approximate that number than to try to identify $f(x)$ as the derivative of another function to allow the integral to be evaluated by the fundamental theorem. We have mentioned the possibility that $f(x)$ is not the derivative of any known function, but there is also the possibility that $f(x)$ itself may not be known. This arises when $f(x)$ is the result of an experiment, allowing individual values to be found, but not relating those values to any function studied in calculus. In these cases, we follow the definition of the integral by Riemann sums and break the interval into small pieces by introducing and increasing sequence x_i for $0 \le i \le n$ with $x_0 = a$

and $x_n = b$. We get nicer formulas if these points are equally spaced, so that $x_{i+1} - x_i = (b-a)/n$, although this is not essential. The integral is written as a sum from $i = 0$ to $n-1$ of $x_{i+1} - x_i$ times a typical value, denoted $f(x_i^*)$ in the Riemann sum, with $x_i \le x_i^* \le x_{i+1}$. For most methods, our typical value is an average of values of the function at certain locations in the interval $[x_i, x_{i+1}]$ which is not necessarily found by first locating x_i^*. If we could get the value that we defined as the average value of the function, that would give the integral exactly (by definition), but a process that attempts to average the function usually leads to a good numerical method.

Composite integration rules. Although the formulas may be rewritten later to combine like terms, these formulas are always derived in a form in which the same construction is applied to each segment from x_i to x_{i+1}. The description uses only the relative position of points in this interval, so it is easy to determine how they are affected by changes of scale. Typical examples are: (the *midpoint rule*) the value of the function at the midpoint of the interval; or (the *trapezoidal rule*) the average of the values at the endpoints. The most elaborate rule described in this course is Simpson's rule, in which these local averages are the sum of $1/6$ of the value at each endpoint and $2/3$ of the value at the midpoint.

If the x_i are equally spaced, the *sum* of these $f(x_i^*)$ is multiplied by $(b-a)/n$ to get the integral — or divided by n to get the average. In the midpoint rule, the *interior* x_i wind up being counted with weight 1 in the sum, $1/2$ as the right endpoint of one interval and $1/2$ as the left endpoint of the next, while a and b have weight $1/2$ each. In Simpson's rule, a and b have weight $1/6$ while points playing the role of midpoints have weight $2/3$ and those that separate the interior intervals have weight $1/3$. Since the midpoints and separating points are equally spaced, formulas are sometimes stated as if all of these points are used as a partition of $[a, b]$. This has the effect of doubling the weights in the form described here. To guard against accidentally doubling or halving the value you compute, it is useful to check formulas by expressing results in terms of averages.

The description of Simpson's rule with equally spaced points x_i for $0 \le i \le 2n$ in terms of the division of $[a, b]$ into n (rather than $2n$) subintervals leads to the formula

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n.$$

Note that this averages two quantities that are aiming to compute the same thing.

Error estimates. First let's give the relation between the error estimate for Taylor's formula and the error using the midpoint rule. We first describe things on a reference interval, and then give the effect of scaling. On the interval $[-1, 1]$, Taylor's formula gives

$$g(u) = g(0) + g'(0)u + \frac{g''(\zeta)}{2}u^2, \qquad (T)$$

where ζ is a value between 0 and u, that is actually a function of u, but we will need only a rough estimate

of its size. Integrate (T) to obtain

$$\int_{-1}^{1} g(u)\,du = \int_{-1}^{1} g(0)\,du + \int_{-1}^{1} g'(0)u\,du + \int_{-1}^{1} \frac{g''\big(\zeta(u)\big)u^2}{2}\,du.$$

The first two terms on the right are $2g(0)$ and 0. In the third term, $u^2/2$ is a function that is always nonnegative, and it is multiplied by $g''\big(\zeta(u)\big)$, for which bounds are known. The integral of this term is between the same bounds times the integral of $u^2/2$ from -1 to 1, which is $(1/6) - (-1/6) = 1/3$.

To apply this to $f(x)$ on the interval $[x_i, x_{i+1}]$, we want $f(x) = g(u)$ and

$$x = \frac{x_i + x_{i+1}}{2} + \frac{x_{i+1} - x_i}{2}u$$

If we differentiate with respect to u, the chain rule gives

$$f'(x)\frac{x_{i+1}-x_i}{2} = g'(u)$$
$$f''(x)\left(\frac{x_{i+1}-x_i}{2}\right)^2 = g''(u)$$

The bounds on $g''\big(\zeta(u)\big)$ in our reference interval $[-1,1]$ are the bounds on $f''(x)$ for $x_i \le x \le x_{i+1}$ multiplied by $\left(\frac{x_{i+1}-x_i}{2}\right)^2$. Thus, our error estimate contains one factor depending on f and one factor proportional to the square of the distance between the x_i. If we are integrating a fixed function over a fixed interval, divided into n parts, the error is proportional to n^{-2}

Interpolation formulas. Newton (and, later, Hermite) gave formulas that approximated a function by a polynomial expressed in terms of values of the function (and its derivative) at a fixed set of points. Taylor's formula is the limiting case of these formulas in which only one point is used, but several derivatives are used. These more general interpolation formulas

are like Taylor's formula in the sense that the error term resembles the next term in the series. Simpson's rule is the result of integrating an interpolation formula based on the endpoints and the center of an interval with the center counted twice.

An example. Consider problem 9 of section 7.8:

$$\int_0^{1/2} \cos(e^x)\,dx \text{ with } n = 8$$

in the trapezoidal rule, midpoint rule and Simpson's rule.