Differential equations. A differential equation is any equation involving an unknown $f(x)$ and its derivatives. This is so general a definition that it is useless. In order to have a theory, the equations must be restricted in some way.

It is more convenient to develop the theory using the Leibniz dy/dx notation rather than in terms of functions. One of the simplest types of equations is one of the form

$$\frac{dy}{dx} = f(x). \qquad (I)$$

That is, y appears only through its derivative with respect to x and we have an expression for that derivative in terms of x. The solution of this equation expresses y as the indefinite integral of $f(x)$. Although indefinite integration is very familiar, the expression of this simple example as a differential equation tells us a great deal. First, we expect a general solution that depends on a parameter (the $+C$ that is added to every indefinite integral); and, second, solution in terms of elementary functions may be impossible, even though it may be possible to guarantee by analytic means that

a solution exists.

Because of these observations, the theory of differential equations concentrates on proving existence of solutions and providing an interpretation of the constant of integration that assures us that a differential equation together with one additional requirement of the solution will have a unique solution.

Where do differential equations come from? Although our brief visit to this subject will stress the ability to solve certain equations using the calculus we know, the interest in these equations arises from their ability to model things that are happening in the real world.

One example is the study of population growth. The simplest model assumes that all effects on the change in population are strictly proportional to the present population. However reasonable this may sound, its expression in formal terms is the differential equation

$$\frac{dP}{dt} = kP \qquad (1)$$

We shall see that this equation can be solved easily, and all solutions that are not constant must either grow rapidly or approach zero as time increases. One attempt to get models that respect limits of growth is to add assumptions to the model so that the differential equation will admit bounded solutions. One common example is the *logistic equation*

$$\frac{dP}{dt} = kP(L - P). \tag{2}$$

It is immediate from the form of this equation that P can only increase if $0 < P < L$. The theory of differential equations will show that a solution that starts in this range will remain there for all time.

Mixing and chemical reactions can be described as analogs of population growth, and will lead to similar differential equations.

Newton's laws of motion lead to other examples. The statement

$$F = m\frac{d^2x}{dt^2}$$

says that if the force F and mass m are known in terms of time t or position x, or some combination

of t and x, there is a differential equation relating x and t. It will turn out that the best way to deal with equations like this involving second derivatives is to invent a new variable to represent dx/dt and write a system of equations using only first derivatives. The physicists have anticipated this by giving this quantity the name "velocity". While we are at the business of looking at systems, we should allow motion in more than one space dimension. The physical properties of all types of mechanical linkages are modeled by equations derived in this way.

The components of electrical circuits are frequently described as analogs of mechanical devices, allowing the properties of electrical circuits to be described by differential equations similar to those governing mechanical linkages.

Initial value problems. The examples suggest that differential equations give a complete description of the behavior of the solution if we know how things start. That is, the equations giving the derivatives of all the dependent variables with respect to the independent variables, supplemented with the value of all variables at one value of the independent variable will always have exactly one solution. There are some technical conditions, but such a result is true. The nature of the solution is also subject to some interpretation. The following result is true, and gives the flavor of the main theoretical result, although it is weaker than the usual statement.

Suppose $dy/dx = f(x, y)$ where $f(x, y)$ has a continuous derivative with respect to x or y for each fixed value of the other variable, and suppose that a point (x_0, y_0) is given. Then for every rectangle containing (x_0, y_0) in its interior, there is a curve through (x_0, y_0) that is the graph of a function $y = g(x)$ satisfying the equation that extends to the boundary of the rectangle.

Note that it is possible that this allows the graph to reach the top or bottom edge before it reaches a side, so the function is not guaranteed to be defined for all values of x that you have allowed. From the point of view of calculus, the restriction on $f(x, y)$ is minor, since the differentiation formulas tell you that a function has a derivative by telling you exactly what that derivative is.

Separable equations. There is one family of equations that we can solve easily, that is more general than those given by (I), including equations (1) and (2) and allowing examples whose solutions have vertical asymptotes. This form is

$$\frac{dy}{dx} = f(x)g(y). \qquad (S)$$

Dividing by $g(y)$ (as long as it isn't zero), we get

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

If $H(y)$ is a function with $H'(y) = 1/g(y)$, the left side is the derivative with respect to x of $H(y)$ whenever y is a function of x. If $F(x)$ is a function with $F'(x) = f(x)$, the right side is the derivative with respect to x of $F(x)$. Since these derivatives are required to be equal, the functions must differ by a constant. Thus all solutions are functions defined implicitly by

$$H(y) = F(x) + C.$$

This result does not lend itself to be summarized by a formula, since it takes too long to define everything that appears in this solution. However, in each individual example, the solution can be found by using a simple description of what we just did. The process is easier if we treat the *differentials* dx and dy as meaningful quantities. The rule is to put everything depending on y on one side and everything depending on x on the other side, arranged so that the expressions we have are multiples of dy or dx; then integrate these expressions. This approach can be justified in terms of integration of functions and the rules for substitution in integrals, but it is not necessary to appeal to this justification every time that we use this method.

While this method is limited to very special equations, there are enough examples to allow some variety. However, you should not use this method if variables in the equation cannot be separated. The study of differential equations is rich enough to be a full semester course just to acquire the skills to solve those equations whose solutions are as easily recognized as the integrals we did earlier in this course.

Examples. In addition to the general equations (1) and (2), we will do the following exercises from Section 8.1:

$$\frac{dy}{dx} = y^2 \qquad \#1$$

$$y\frac{dy}{dx} = x \qquad \#3$$

$$x^2\frac{dy}{dx} + y = 0 \qquad \#5$$

$$\frac{du}{dt} = e^{u+2t} \qquad \#7$$

$$xe^{-t}\frac{dx}{dt} = t \quad x(0) = 1 \qquad \#11$$

Direction fields. If you are interested in solving an equation in some fixed rectangle, it is often useful to draw a picture that describes the equation geometrically. When y is a function of x, the value of dy/dx is the slope of the tangent at a point of the curve. The differential equation gives an expression for this as a function of *both* coordinates of the point. A picture showing a short segment of this slope at a grid of points is called a *direction field* of the equation. The solution curves have the property that they have these lines as tangents. Often, one can get a good idea of properties of the solutions just by looking at these pictures.

When we were describing the solution of (S), we noted that it was necessary to assume that $g(y) \neq 0$. We can now describe that excluded case. If $g(y_1) = 0$, then $dy/dx = 0$ everywhere on the line $y = y_1$, so the direction field is horizontal on this line. This means that the direction field points along this line and this constant function is a solution of the equation. If all initial value problems have unique solutions, no two solution curves can have a common point, so all other solutions will remain on one side of this line.