One problem with many features. Problem 7.8.4 asks to approximate

$$\int_0^1 \cos(x^2)\,dx.$$

We first consider some numerical approximations, then we ask how good they are. This will lead us to considerations of Taylor series, which also provide an alternative approach to the numerical evaluation of the integral.

Since the interval of integration is of length 1, the integral is equal to the average value of the function. All of the numerical methods will give us weighted averages of some finite set of function values.

The simplest rules are the midpoint rule with $n = 1$, denoted M_1 and the trapezoidal rule with $n = 1$, denoted T_1. In T_1, the average of the function is estimated by the average of the values at the endpoints. We write $j(x) = \cos(x^2)$. Then

$$j(0) = 1 \quad j(1) = 0.5403023058681$$

and T_1 is the average of these values, so

$$T_1 = 0.7701511529341.$$

The value of M_1 is easier to find. It is just the value of $j(x)$ at the midpoint of the interval. Thus

$$M_1 = j(0.5) = 0.9689124217106$$

If the step size is cut in half, the new rules will involve the values used in the old rules, except for the midpoint rule that will introduce an entirely new sequence of values. The formulas for the trapezoidal rule and Simpson's rule are

$$T_{2n} = (M_n + T_n)/2$$
$$S_{2n} = (2 * M_n + T_n)/3$$

This is enough to give *immediately*

$$T_2 = 0.8695317873224 \quad S_2 = 0.9026586654518$$

To find M_2, we average

$$j(0.25) = 0.9980475107001$$
$$j(0.75) = 0.8459244992311$$

to get

$$M_2 = 0.9219860049656$$

which gives

$$T_4 = 0.895758896144 \quad S_4 = 0.9045012657512$$

Continuing in this way, the successive values of Simpson's rule are

$$S_8 = 0.904524159207$$
$$S_{16} = 0.9045242525675$$
$$S_{32} = 0.9045242391159$$
$$S_{64} = 0.9045242379809$$
$$S_{128} = 0.9045242379054$$

The error estimates for these rules in this case are

$$E_T \le K_2/(12n^2)$$
$$E_M \le K_2/(24n^2)$$
$$E_S \le K_4/(180n^4)$$

where K_k is a bound on the absolute value of the k^{th} derivative of $j(x)$ on the interval of integration. One can get a reasonable expression for $j''(x)$ and use it to show that $K_2 < 4$. However, it is not easy to get a useful expression for $j^{(4)}(x)$. The value of K_4 will turn out to be around 40, confirming the observation that this rule is better than the trapezoidal rule for $n = 2$ and much better for larger values.

However, you really should not need to pipe the output of a symbolic differentiation program into a graphing program just to estimate an error term. An alternative that works well in this case is to find the Taylor series of $j(x)$. Since $j(x)$ is defined by substituting a power of x into a function whose Taylor series we know, we get

$$j(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}.$$

One can also substitute x^2 into the Taylor series error estimate for $\cos x$ to show that this series converges to $j(x)$ for all x and to estimate the error. Once we know that the series converges, we can differentiate to get series for derivatives of $j(x)$. In particular,

$$j^{(4)}(x) = -12 + 70x^4 - \frac{33}{2}x^8 + \dots$$

For $0 < x < 1$, this is seen to be an alternating series whose terms decrease to zero, so we can use the alternating series error estimate to bound the function.

We can also integrate the Taylor series. Term by term integration of a power series, using a zero constant term, gives a function whose value at zero is zero, and whose derivative is the function represented by the original series.

Thus

$$\int_0^x j(t)\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(4n+1)\cdot\big((2n)!\big)}$$

The value at 1 is

$$\begin{aligned} 1 - 0.1 + 0.00462962962963 & \\ - 0.0001068376068376 & \\ + 0.000001458916900093 & \\ - 0.0000000131225329638 & \\ + 0.00000000008350702795147 & \\ - \dots = 0.9045242379003. & \end{aligned}$$

The next term is bounded by 4×10^{-13} in absolute value. Since this is an alternating series whose terms decrease monotonically to zero, the error is bounded by the next term in the series. In this case, half a dozen terms of the series give more accuracy than Simpson's rule combining 64 values of the function. Moreover, having the function at your fingertips obscures the fact that each function evaluation uses something like the Taylor series for $j(x)$, so we would need half a dozen terms of that series for each evaluation of $j(x)$. Thus, the series isn't just ten times as efficient as numerical integration in this case — it is about 100 times as efficient.

On the other hand, it is easy to construct examples for which numerical integration is more efficient than a Taylor series. The general binomial expansion gives a series for

$$k(x) = (1 + ax^n)^r$$

where n is an integer and r is an arbitrary real number. Such expressions with $r = 1/2$ appear in arc-length integrals. Although the series for $k(x)$ is easy to generate, it only converges for $|ax^n| < 1$. If you need to integrate beyond this interval, it would be necessary to use Taylor series expansions based on different points. This begins to resemble the evaluation of the function at points throughout the interval of integration that appears in numerical integration methods.