Parametric equations. Up until now, we have talked about curves as the graphs of functions, with $y = f(x)$; or sometimes with some test for a point to be on the curve, like $x^2 + y^2 = 1$ for the unit circle. These approaches look at the curve after the ink has dried, and they ignore all considerations of how the curve was drawn. We now shift the emphasis. The process of drawing a curve involves indicating where the pen is at each time. Since location in the plane is given by x and y coordinates, this says that we should give x and y as functions of t (representing time). A simple example, but one which is very important is the circle. If the center is at (a, b) and radius is r, it is common to draw the circle by starting somewhere (e.g. at the point *due East* of the center) and move in some direction around the center at constant speed, so that the angle is proportional to t. The simplest example is

$$x = a + r\cos t \quad y = b + r\sin t.$$

Those of you who continue to *Multivariable Calculus* will see, at the end of Chapter 11, that some things

are nicer if you can draw the curve at constant speed — essentially using the arc length s as a parameter — but this is rarely possible.

The previous study of graphs of functions can be included in the parametric approach by taking x as the parameter. There are also some examples given in the text where a parameter is chosen having nothing particular to recommend it. We will try to concentrate on examples in which the parameter makes a *useful* contribution.

Cycloids. The curves obtained by following a point on a circle that rolls *without slipping* on another curve — typically a line or another circle — are called *cycloids*. They probably first attracted attention because they had a description that led to an easy way to design a linkage to draw them. The introduction of coordinates then leads to simple parametric equations for these curves. They turned out to have interesting mathematical properties so we continue to encourage people to draw them and admire them.

The phrase “without slipping” is taken to mean that the distance along the curve supporting the rolling circle should equal the length of the arc of the part of the circle that has touched this curve. Thus, if a circle of radius r has rolled through an angle t along the x-axis, its center has moved a distance of rt. If we follow a point that starts at the origin, its original direction is directly below the center, and as the circle moves to the right, the direction from the center to the point we are following moves clockwise. At time t, the center of the circle is at (rt, r) and going from the center to the location of the pen is at an angle t clockwise from *due South*, which subtracts $r\sin t$ from x and subtracts $r\cos t$ from y. The result is

$$x = rt - r\sin t \quad y = r - r\cos t.$$

If the circle of radius r rolls on the outside of a circle of radius R centered at the origin, and we follow a point that starts at $(R, 0)$, then as the center moves through an angle t (counterclockwise) to

$$\big((R+r)\cos t, (R+r)\sin t\big),$$

touching an arc on the fixed circle of length Rt. This means that one must follow an arc of length Rt, and

hence angle Rt/r, counterclockwise from the direction towards the origin. The total angle clockwise from *due West* is $(R+r)t/r$. This subtracts r times the cosine of this angle from x and subtracts r times the sine from y. The equations become

$$x = (R+r)\cos t - r\cos\left(\frac{(R+r)t}{r}\right)$$
$$y = (R+r)\sin t - r\sin\left(\frac{(R+r)t}{r}\right)$$

Exercises. Here are some exercises from Section 9.1.

$$x = 1 - t \quad y = 2 + 3t \qquad \#1$$
$$x = 3t^2 \quad y = 2 + 5t \quad 0 \le t \le 2 \qquad \#3$$
$$x = \cos^2\theta \quad y = \sin^2\theta \qquad \#9$$
$$x = e^t \quad y = e^{-t} \qquad \#11$$

Tangents. In differential calculus, the differentials dx and dy are sometimes used as local coordinates giving an equation of the tangent line. That is, if

(a, b) is a point on the graph of $y = f(x)$ (so that $b = f(a)$), the formal equation $dy = f'(x)\,dx$ is transformed into an equation of the tangent line to this graph at (a, b) by substituting a for x, b for y, $x - a$ for dx and $y - b$ for dy.

When we studied *implicit functions*, we were able to use this notation in a formal way to extend results to curves defined by an equation relating x and y that could not easily be solved to give y as a function of x. In these examples, all we needed was a good reason to believe that y could be written a function $f(x)$ somewhere near a point (a, b) on the curve. Differentiating the equation defining the curve, and evaluating at (a, b) led to an equation in a, b and $f'(a)$ that was always linear in $f'(a)$. The main theoretical result assures us that this gives a correct answer whenever it gives any answer at all (i.e., whenever the coefficient of $f'(a)$ isn't zero).

Working with parametric equations also uses this approach of pretending that there is a function such the

$y(t) = f\big(x(t)\big)$. In this case, the chain rule gives

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}.$$

Since dx/dt and dy/dt can be found as functions of t, we get a result the can be written

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

With t used to identify points on the curve this equation allows us to write the equation of the tangent line at any point of the curve that we can identify.

Involutes. A family of curves vaguely related to cycloids can be described by saying that you draw the curve using a pen that is at the end of a string being unwrapped from some curve, usually a circle. After a distance t has been unwrapped, the pen is at a distance t along the tangent line to the curve at a point t

units of arc length from the starting point. For a unit circle, arc length is angle, so the point of tangency is $(\cos t, \sin t)$ and the tangent at this point is given by

$$\frac{y - \sin t}{x - \cos t} = \frac{\cos t}{-\sin t}$$

To go back along this tangent a distance of t adds $t \sin t$ to x and subtracts $t \cos t$ from y.

More Exercises. Here are some exercises from Section 9.2 in which you are asked to find the equation of the tangent line.

$$x = t^2 + t \quad y = t^2 - t \quad t = 0 \qquad \#1$$

$$x = 2t + 3 \quad y = t^2 + 2t \quad (5, 3) \qquad \#5$$

$$x = 2 \sin 2t \quad y = 2 \sin t \quad (\sqrt{3}, 1) \qquad \#7$$