Arc length. The arclength integral, written in the form

$$\int \sqrt{dx^2 + dy^2}$$

suggests that the length of a parametrized curve is given by

$$\int \sqrt{x'(t)^2 + y'(t)^2}\, dt. \qquad (L)$$

As before, limits of integration should be the values of t that describe the endpoints of the arc being measured. As long as this integral makes sense, its Riemann sums can be shown to approximate the length of the curve. A Riemann sum of the integral corresponds to dividing the parameter interval into subintervals, and substituting the endpoints of these subintervals into the description of the curve, divides the curve into small arcs. The textbook points out that a direct use of the mean value theorem gives an expression that is a little more general than a Riemann sum for this integral, but these sums can be shown to converge to (L) whenever $x(t)$ and $y(t)$ have continuous derivatives.

From the point of view of Calculus, these technical restrictions aren't much of an obstacle. You only work with functions for which you have a formula for the derivative — usually in terms of other functions that appear in differentiation formulas. Since differentiability implies continuity, these technical conditions were verified in obtaining the formulas. Once in a while, one of these formulas will introduce something that needs to be watched: for example, although $x^{1/3}$ can be defined for all real x, its derivative doesn't exist when $x = 0$. The concept of *improper integral* was invented to allow us to work with these cases, reassuring us when we are able to get an answer, and providing an explanation of the difficulty when things don't work.

Examples. Examples given in the text are the circle

$$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi,$$

and the cycloid

$$x = r(t - \sin t) \quad y = r(1 - \cos t) \quad 0 \le t \le 2\pi.$$

Several other curves appear in the exercises. Generally, only special curves lead to arc length integrals that can be evaluated in closed form, so a few examples will exhaust all that you are likely to meet in a course. In the real world, you may meet other curves whose length is needed; for these, the knowledge that length is given by an integral can be used to obtain a numerical approximation.

Here are some exercises

$$x = t^3 \quad y = t^2 \quad 0 \le t \le 4 \qquad \#5$$
$$x = 2 - 3\sin^2 t \quad y = \cos 2t \quad 0 \le t \le 2\pi \qquad \#7$$
$$x = e^t \cos t \quad e^t \sin t \quad 0 \le t \le \pi \qquad \#9$$

More difficult examples. Some problems that require computer assistance are #17 and #18. Here are some of the details.

The curve in #17 is an *epitrochoid*. This is something like an epicycloid, except that the pen is assumed to be firmly attached to a point on the rolling circle that is not necessarily on the rim. If it is at a distance b

from the center of the circle, a derivation just like the one for the epicycloid gives

$$x = (R + r)\cos t - b\cos\left(\frac{(R + r)t}{r}\right)$$
$$y = (R + r)\sin t - b\sin\left(\frac{(R + r)t}{r}\right)$$

The equation given in the exercise fits this form if $R + r = 11$, $r = 2$ and $b = 4$. Thus, a circle of radius 2 is rolling on a circle of radius 9, with a pen located 4 units from the center of the circle. A complete picture requires t to go through an interval of 4π, since it takes 2π to bring the center of the circle back to its starting point, but the pen is then on the opposite side of the center. When I tried to compute the arclength numerically in Maple with this length, the computation went on for a long time without giving an answer. Thus, something more is needed.

A sketch of the graph shows what appears to be nine equal loops, although a closer look shows that the curve doesn't flow from one loop to the next, but to the second loop in order. That is, adding $4\pi/9$ to t has the effect of rotating through $4\pi/9$. The reflection $t \mapsto -t$ also has the effect of reflecting in the x axis. This means that the interval $0 \le t \le 2\pi/9$ gives an arc that is $1/18$ of the curve. This arc length integral was done quickly by Maple, giving an answer of 16.33486949.

Problem 18 deals with a curve known as the *Cornu spiral*. The parameterization is

$$x = \int_0^t \cos(\pi u^2/2)\,du \quad y = \int_0^t \sin(\pi u^2/2)\,du.$$

Although these parameterizing functions look difficult to work with, you get a pleasant surprise when you set up the arc length integral.