Math 152:10–12 — Fall 1999
TF3 CHM-201
Prof. Bumby

Workshop 4, Textbook Review chapter 7 and preview chapter 10

1. Here are some stupid integration by parts tricks.

(a) Use the formula

$$\int u\,dv = uv - \int v\,du \qquad (P)$$

with $u = x^a$ and $dv = x^b\,dx$ to get a formula for

$$\int x^{a+b}\,dx$$

Try to make this help evaluate the integral, although it really shouldn't tell you anything new.

(b) Now take $u = x^a + 1$ while keeping $dv = x^b\,dx$. Write the integration by parts formula (P) in this case, and apply it to evaluating

$$\int (x^a + 1)x^b\,dx.$$

The result should not be surprising except for its ability to get a correct answer to a simple problem by a difficult method.

2. Consider

$$\int_0^1 \ln x\,dx.$$

The integration by parts formula (P) with $u = \ln x$, $dv = dx$ can be used to evaluate the integral. However, the answer involves $x \ln x$, which cannot be evaluated at $x = 0$. This is not surprising, since the the behavior of $\ln x$ at $x = 0$ makes the original integral improper.

(a) Show that the integral can be evaluated as an improper integral if $\lim_{x\to 0} x \ln x$ exists.

(b) Graph $y = x \ln x$ for $0 < x \le 1$, noting that the function is negative throughout this interval. Also show that the function also has a minimum somewhere in this interval (you should be able to locate the minimum without being told anything more about it), and is decreasing for x less than the location of the minimum and increasing for larger x.

(c) Since $x \ln x$ is bounded and decreasing, the limit as $x \to 0$ through positive values exists. This property also assures us that the limit of the function is the same as the limit of the sequence of values of the function at 2^{-n}.

(d) It is not difficult to show that the value at 2^{-n-1} is greater than c times the value at 2^{-n} for some c strictly between 0 and 1. Interpret this statement, and show that forces the limit to be zero. (Since the function is negative for these values, this inequality says that the distance to zero is shrinking by a factor of c).

. . . continued on other side

3. As part of showing that

$$\int_0^\infty \frac{1}{x}\,dx$$

diverges, an exact computation of

$$I_n = \int_{2^n}^{2^{n+1}} \frac{1}{x}\,dx$$

shows that it is $\ln 2$ for all n. This shows that the *average* of the function $1/x$ on this interval is $(\ln 2)/2^n$.

(a) Using only the fact that an average of a function on an interval is always between the smallest and largest values of the function on the interval, produce upper and lower bounds on the average that are only slightly weaker than the exact value.

(b) Show that the same bounds also apply to the average of the 2^n numbers $1/k$ for $2^n < k \leq 2^{n+1}$.

(c) Show that $\sum_{n=1}^{\infty} 1/n$ diverges.

4. A common way to define a sequence uses a formula

$$a_{n+1} = f(a_n) \qquad (F)$$

together with an initial value a_0. For continuous functions $f(x)$, the definition of limit implies that the limit L of such a process, if it exists, satisfies $f(L) = L$. This allows approximate solution of equations by suitable iterations.

(a) Take $f(x) = \cos x$. Sketch the graph of the function and use the sketch to show that there is only one solution to $f(x) = x$.

(b) Construct two sequences defined by (F) with this $f(x)$ and different x_0. They should appear to converge to the same value.

(c) Instead of iterating $f(x) = \cos x$, iterate the inverse function $f^{-1}(x) = \arccos x$. What do these sequences look like? Do they converge?

(d) Repeat the previous steps for $f(x) = 2\cos x$. You should find that there is still a unique solution to $f(x) = x$, but it cannot be found by iterating f. Does the iteration of f^{-1}, starting from $x_0 = 0$, lead to a convergent sequence in this case?

End workshop 4