Math 152, Sections 5, 6, 7, Spring 1999, Week 5

1. Consider the following improper integrals:

$$(1)\ \int_1^\infty \frac{\ln x}{x}\,dx \qquad (2)\ \int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx \qquad (3)\ \int_1^\infty \frac{\ln x}{x^3}\,dx$$

a) Graph the integrands and determine which integrals are larger than the others.

b) Do integrals (1) and (2) converge or diverge? (*Hint:* Try a substitution to evaluate one of the integrals.)

c) Does your knowledge help you decide whether integral (3) converges?

d) Explain why $\dfrac{\ln x}{x^3} < \dfrac{1}{x^2}$ for all $x > 1$, and use this to decide whether integral (3) converges.

2. Assume that a is a positive constant. Let R be the region bounded above by $y = 1/x^a$, below by $y = 0$, and on the left by the line $x = 1$.

a) Sketch the curves $y = 1/x^a$ for $a = .5$, 1 and 2. Which hugs the x-axis the closest?
b) For which values of a do you get a convergent integral when you attempt to calculate the area of R?
c) Same as b), but for the volume of the solid obtained by rotating R around the x-axis.
d) Same as c), but for the volume of the solid obtained by rotating R around the y-axis.

Note: The following problems on numerical integration expect that you will use the program `NUMINT`, which was given on a separate handout and which you should have already keyed into your calculator.

3. This problem concerns the system error in the Trapezoidal Rule* (as opposed to round-off error, which you should ignore in this problem). Suppose that you decide to compute the decimal expansion of π by the formula

$$\pi = \int_0^1 \frac{4}{1+x^2}\,dx,$$

using the Trapezoidal Rule to approximate the integral.

a) Let $f(x) = 4/(1+x^2)$. Show that $|f''(x)| \le 8$ for all x in the interval $[0,1]$.

b) Estimate the error if you use the Trapezoidal Rule with $n = 10$. (Your answer should give a specific number such that the error is no bigger than that number.)

c) Use `NUMINT` to compute the integral with the Trapezoidal Rule with $n = 10$. Does this fit with your answer to b)?

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* If you'd rather use Simpson's rule, well, go ahead, but in part (a) you'll have to estimate $|f^{(4)}(x)|$ instead of just $|f''(x)|$.

d) If you wanted 100 place accuracy, how large a value of n would you have to use to be confident of that accuracy?

4. Let $a \geq 1$. Use a substitution to calculate $\int_a^\infty xe^{-x^2}\,dx$. Now use this result to show that $\int_a^\infty e^{-x^2}\,dx \leq \frac{1}{2}e^{-a^2}$.

b) Calculate an approximate value for the convergent improper integral $\int_0^\infty e^{-x^2}\,dx$ as follows: Find a value of a so that $\frac{1}{2}e^{-a^2} < 10^{-5}$. Now calculate $\int_0^a e^{-x^2}\,dx$ approximately using `NUMINT`. (The exact value of the integral, obtained by methods of multivariable calculus, is

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2};$$

this constant is important in probability and statistics.)

c) Discuss all the sources of error in your calculation.

d) Using your answer to b) calculate a numerical value for $\int_0^\infty x^2e^{-x^2}\,dx$, (*Hint:* Write the integrand as $x(xe^{-x^2})$, and use integration by parts.)