REVIEW PROBLEMS FOR FINAL EXAM
MATH 151, SPRING 2008

1. Prove that $x^2 + 3x + 3 \geq 0$ for all x.

2. Solve $\sin 2x + \cos x = 0$ for $0 \leq x \leq 2\pi$.

3. Find the inverse function of $f(x) = \dfrac{x-2}{x-1}$ and determine its domain and range.

4. Evaluate the following limits if possible or state that it does not exist.

(a) $\displaystyle\lim_{x\to 0} \frac{x^3}{\sin x - x}$. (b) $\displaystyle\lim_{n\to\infty} \left(\frac{n+3}{n}\right)^n$. (c) $\displaystyle\lim_{x\to\infty} \frac{\ln(e^t+1)}{t}$.

(d) $\displaystyle\lim_{x\to 0} \sqrt{x}\ln x$. (e) $\displaystyle\lim_{x\to 4} \left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right)$. (f) $\displaystyle\lim_{x\to\infty} \frac{12x+1}{\sqrt{4x^2+4x}}$.

(g) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$. (h) $\displaystyle\lim_{x\to -1} \frac{3x^2+4x+1}{x+1}$. (i) $\displaystyle\lim_{x\to 8^-} \frac{|x-8|}{x-8}$.

5. Suppose that f is a differentiable function with $f(1) = 2, f'(1) = 4$ and $g(x) = x^4 - x + 1$. Use this information to calculate:
(a) $(fg)'(1)$. (b) $(f/g)'(1)$. (c) $f \circ g)(0)$. (d) $(f \circ g)'(0)$. (e) $(f \circ g)'(1)$.

6. With A and B constants, a function f is defined by $f(x) = \begin{cases} \dfrac{1}{x} + A & x < -1 \\ |x| & -1 \leq x \leq 1 \\ \dfrac{1}{x} + B & 1 < x \end{cases}$.

(a) Find A and B so that f is continuous everywhere.
(b) Sketch the graph of $f(x)$.
(c) At which points does f fail to be differentiable?

7. Let $f(x) = 2\sin x + \sin^2 x$.
(a) Find $f'(x)$.
(b) Find an equation of the tangent line to the graph of $f(x)$ at $x = \pi$.
(c) Find all values of x for which the tangent line is horizontal.

8. Find the derivative of the following functions:

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(a) $f(x) = \tan(x + e^x)$. (b) $f(x) = \dfrac{e^{x^2}}{1 + \sin 2x}$. (c) $h(y) = (y^2 + \sin(2y))\cos(2y - 1)$.

(d) $y = \tan(\cos(3x - 1))$. (e) $y = (x + \frac{1}{x})^3$. (f) $y = \sqrt{\dfrac{x(x+2)}{(2x+1)(3x-2)}}$.

(g) $\tan(x^2 y) = x + y$. (h) $f(z) = \cos^{-1}\left(\dfrac{z}{1 - z^2}\right)$. (i) $y = x^{\tan^{-1} x}$.

(k) $y = \dfrac{e^x \cos^{-1} x}{\ln x}$. (l) $\displaystyle\int_0^x \sin^{-1}\sqrt{t^2 - 2t}\,dt$. (m) $\displaystyle\int_{x^2}^5 \frac{\ln(7-t)}{e^t - 2}\,dt$.

9. Prove rigorously that $\displaystyle\lim_{x \to 5} \frac{1}{x} = \frac{1}{5}$.

10. Let $s(t) = \dfrac{t^3}{3} - \dfrac{5t^2}{2} + 6t$ be the position function of some moving particle.
(a) Find the velocity function.
(b) Determine where the particle stop.
(c) Find when the particle is moving forward and backsword.
(d) Find the distance the particle travels during the first 4 seconds.

11. Find all points (a, b) on the parabola $y = x^2 - x$ so that the tangent line to the parabola at the point (a, b) contains point $(2, 1)$.

12. Explain why $x^3 + x - 1$ has a real root in $(0, 1)$.

13. Suppose that u is a positive real number and a, b, c are real numbers so that $\log_u(5) = a$, $\log_u(27) = b$ and $\log_u(32) = c$. What is the numerical value of $u^{2a + \frac{1}{3}b - \frac{2}{5}c}$?

14 Find equations of the tangent lines to the curve $y = \dfrac{x-1}{x+1}$ that are parallel to the line $x - 2y = 2$.

15. Find the maxima and minima for the given functions and the intervals.

(a) $y = \dfrac{1 - x}{x^2 + 3x}$ in $[1, 4]$. (b) $y = |3x^2 - 9|$ in $[-4, 5]$.

16. Sketch the graphs of the following functions (with increasing/decreasing, maxima/minima, concavity/points of inflections, and horizontal/vertical asymptotes):

(a) $y = 4 - 2x^2 + \frac{1}{6}x^4$. (b) $f(x) = \dfrac{1}{x^2 - 1}$. (c) $f(x) = x^2 e^{-x}$.

17. Estimate $\sqrt{26} - \sqrt{25}$ using the linear approximation, and find an error using a calculator.

18. Find all points on the folium $x^3 + y^3 = 3xy$ at which the tangent line is horizontal.

19. Find the equations of the tangent lines of functions at given point or given slope of a tangent line:

(a) $ye^x + xe^y = 4$, $P = (4, 0)$. (b) $f(x) = x^3 - 3x^2 + x + 4$, slope $= 10$.

20. Use the Mean value theorem to prove that $\sin x - \cos x = 3x$ has a solution, and use Rolle's theorem to show that this solution is unique.

21. In the right triangle $\triangle ABC$, the right angle is at C and the legs are $|AC| = 2$ and $|BC| = 6$. A rectangle is to placed inside the triangle, with one corner at C and the opposite corner on the hypotenuse. What are the dimensions of the rectangle that has the largest area?

22. Find the equation of the tangent line to the curve defined by the equation $\ln(xy) + 2x - y + 1 = 0$ at point $(\frac{1}{2}, 2)$.

23. Suppose that $S(x) = \sqrt{x}$ for $x \geq 0$, and let f and g are differentiable functions about which the following is known:

$$f(3) = 2, \qquad f'(3) = 7 \qquad g(3) = 4 \qquad g'(3) = 5.$$

Compute the following:

$$(f+g)'(3) \qquad (f \cdot g)'(3) \qquad \left(\frac{f}{g}\right)'(3) \qquad (S \circ g)'(3) \qquad \frac{f \cdot g}{f - g}.$$

24. Suppose $f''(x) = -3x + \cos(\pi x)$ and $f(1) = 2$ and $f'(1) = -1$. What is $f(5)$?

25. A ladder is leaning against a wall when its base begins to slide along the floor, away from the wall. By the time the base is 12 ft. away from the wall, the base is moving at the rate of 5 ft/sec. How fast is the top of the ladder sliding down the wall then? How fast is the area of the triangle formed by ladder, wall, and floor changing at that time?

26. Sketch the graph of $f(x)$ which satisfies the following conditions: $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$, $f''(x) < 0$ if $-2 < x < 0$, inflection point $(0, 1)$.

27. Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{x}{x^2 + 1}$ in $[0, 2]$.

28. Find the most general antiderivative of the functions:

a) $f(x) = 1 - x^3 + 5x^5 - 3x^7$. b) $g(x) = \dfrac{5 - 4x^3 + 3x^6}{x^6}$

29. Evaluate the following indefinite and definite integrals.

(a) $\int \sqrt{x}(x^2 - 1)dx$ (b) $\int \cos(3 - 4t)dt$ (c) $\int \dfrac{x^2 + 2x - 3}{x^4}dx$.

(d) $\int (4\theta + \sin 8\theta)d\theta$ (e) $\int_0^2 y^2\sqrt{1 + y^3}dy$. (f) $\int_1^5 \dfrac{dt}{(t-4)^2}$.

g) $\int_0^1 t^2 \cos(t^3)dt$. h) $\int_1^8 t^2\sqrt{t+8}dt$.

30. If $\int_1^5 f(x)dx = 12$ and $\int_4^5 f(x)dx = 3.6$, find $\int_1^4 f(x)dx$.

31. Find the area bounded by two curves:

a) $y_1 = 2x^2$ and $y_2 = 8x$. b) $y_1 = \sin x$ and $y_2 = \cos x$, $0 \le x \le \dfrac{\pi}{2}$.

32. (a) Evaluate the limit $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \sin(\dfrac{i\pi}{n})\dfrac{\pi}{n}$.

(b) Find the value of the definite integral: $\int_{-9}^{9} \sqrt{81 - x^2}dx$.

33. a) Use $\ln x = \int_1^x \dfrac{1}{t}dt$ to prove that $\ln 2 < 1 < \ln 3$,

b) Use a) to deduce that $2 < e < 3$.