Math 151, Fall 2001: Exam #2 Review Problems

IMPORTANT NOTE: These problems cover material from Sections 3.5–3.8, 3.10, 3.11, 4.1–4.7, 4.9–4.10, 5.1 and Appendix E. Depending on your section, your exam may cover somewhat more or less. Be sure to check with your individual instructor exactly which sections will be covered on **your** *exam.*

1. A spotlight on the ground shines on a wall 20 meters away. A woman 2 meters tall walks from the spotlight to the wall, at a speed of 0.4 m/sec; her path is perpendicular to the wall. Let x be the distance from her feet to the spotlight and let h be the height of her shadow on the wall. Also let θ be the angle of elevation at the spotlight from the horizontal to the top of her head.

(a) Draw a sketch of the problem and find a formula relating h and x.

(b) When the woman is 4 meters from the wall, find the height of her shadow and the rate of change of the height of her shadow.

(c) What is the rate of change of θ at this moment?

2. Verify that the equation $5x = e^x$ has a solution in the interval $[0, 2]$.

(a) In order to obtain approximate solutions by Newton's method, how must you rewrite the equation?

(b) Write out the recursion equation for Newton's method applied to this particular equation. If Newton's method is applied with an initial guess of $x = 0$, what are the next two approximations to the solution of the equation?

(c) Show graphically how these approximations arise.

3. Suppose that $y = y(x)$ satisfies the equation $2xy - \ln y = 4$ and the condition $y(2) = 1$.

(a) Calculate $y'(2)$ and find the linear approximation $L(x)$ for $y(x)$ around $x = 2$.

(b) Obtain an equation relating dy and dx near $x = 2$.

(c) Using either a) or b), get an approximate value of $y(2.1)$.

(d) What is the concavity of the graph of $y = y(x)$ near $x = 2$?

(e) Is the linear approximation you found for $y(2.1)$ greater or smaller than the true value? Explain in terms of your answer to d).

4. Compute $\displaystyle\lim_{x\to 0} \frac{1-\cos 3x}{x^2}$, $\displaystyle\lim_{n\to\infty} \left(\frac{n+3}{n}\right)^n$, $\displaystyle\lim_{x\to\infty} \frac{\arctan x}{2x}$, and $\displaystyle\lim_{x\to\infty} \left[\frac{1}{2}\ln\left(x^2+1\right) - \ln\left(\frac{x}{2}\right)\right]$.

5. a) Suppose that $f(x)$ is a differentiable function and you know the following:

$$f(1) = 11 \text{ and } -2 \leq f'(x) \leq 3 \text{ for } 0 \leq x \leq 4.$$

Use the Mean Value Theorem to find numbers A and B so that

$$A \leq f(1.3) \leq B$$

Explain how you got your answer.

b) Let $f(x) = \cos(10x^2)$. Use the Mean Value Theorem to find a number M with the property

$$|f(b) - f(a)| \leq M|b - a| \quad \text{for all numbers } a, b \text{ with } 0 \leq a \leq b \leq 2.$$

6. Suppose f is a function defined for all x. Assume also that f' and f'' exist for all x. (This means, in particular, that the graph of f has no corners.) Assume that:

$$f(0) = -2, f(2) = 2, f(4) = 1$$

$$f'(0) = f'(2) = f'(4) = 0$$

$$f''(-3) = f''(-1) = f''(1) = f''(3) = 0$$

$$f'(x) < 0 \text{ for } x < 0 \text{ and for } 2 < x < 4$$

$$f'(x) > 0 \text{ for } 0 < x < 2$$

$$f''(x) < 0 \text{ for } x < -3, \text{ for } -3 < x < -1 \text{ and for } 1 < x < 3$$

$$f''(x) > 0 \text{ for } -1 < x < 1 \text{ and for } 3 < x$$

a) Find all local minima and maxima inside the interval $-4 < x < 4$.

b) Find all inflection points.

c) Sketch the part of the graph of $y = f(x)$ that lies over the interval $-4 \leq x \leq 4$.

7. For each of the following functions $f(x)$, find all asymptotes of the graph of f; all critical points of f; all local maxima and minima of f; all inflection points of f; the intervals where f is increasing, decreasing, concave up, and concave down. Sketch the graph.

$f(x) = \dfrac{2x+3}{x+1}$ $\qquad f(x) = x^4 - 8x^3 + 18x^2$ $\qquad f(x) = \arctan x + \ln(1 + x^2)$

8. For each of the three functions in the previous question, find the absolute maximum and absolute minimum of f on the interval $[-3, 3]$.

9. In the right triangle ΔABC, the right angle is at C and the legs are $|AC| = 4$ and $|BC| = 12$. A rectangle is to be placed inside the triangle, with one corner at C and the opposite corner on the hypotenuse. What are the dimensions and area of the largest rectangle that fits?

10. If $f'(x) = 3\cos 2x + 4\sin x - x + 1$ for all x, and if $f(0) = 5$, then what is $f(x)$? Check your answer by differentiation.

11. Express the area of the region of the plane which lies under the graph of $3x^2 - 2x$ for $1 \leq x \leq 2$ as the limit of a sum of areas of approximating rectangles, using right-hand endpoints of the partitioning intervals as sample points. Find the area by evaluating the limit.