Solutions to Exam I, Math 154, Spring 2003

1. The region R extends from $x = 1$ to $x = 3$ because the two curves intersect for these two x-values.

a) To get the volume when R is revolved around the x-axis, use the washer method with inner radius $r_{in} = x - 1$ and outer radius $r_{out} = (x-1)(4-x)$. The volume is

$$V = \pi \int_1^3 \left[(x-1)^2(4-x)^2 - (x-1)^2\right] dx.$$

b) To get the volume when R is revolved around the axis $y = -1$, use the washer method again, but this time add 1 to the inner and outer radii appearing in part a).

$$V = \pi \int_1^3 \left[((x-1)(4-x)+1)^2 - x^2\right] dx.$$

c) To get the volume when R is revolved around $x = -2$, it is easiest to use the cylindrical shell method. At each x in $[1, 3]$, the radius of the shell of revolution is equal to $x + 2$ and its height is $(x-1)(4-x) - x$. Thus,

$$V = 2\pi \int_1^3 (x+2)\left[(x-1)(4-x) - x\right] dx.$$

2. When evaluating indefinite integrals, don't forget the constant of integration c !

a) Use the substitution $u = x^4$. Then

$$\int x^3 e^{x^4}\, dx = \frac{1}{4}\int e^u\, du = \frac{1}{4}e^u + c = \frac{e^{x^4}}{4} + c$$

b) Use the identity $\sin^2 x = 1 - \cos^2 x$, and then the substitution $u = \cos x$. Then, omitting the work of the substitution step,

$$\int \sin^3\theta\cos^2\theta\, d\theta = \int \sin\theta(1-\cos^2\theta)\cos^2\theta\, d\theta = \frac{\cos^2\theta}{5} - \frac{\cos^3\theta}{3} + c$$

c) Integrate by parts, taking $u = \ln t$, $du = (1/t)\,dt$, and $dv = \sqrt{t}\,dt$, $v = (2/3)t^{3/2}$.

$$\int \sqrt{t}\ln t\, dt = \frac{2}{3}t^{3/2}\ln t - \int \frac{2}{3}t^{1/2} = \frac{2}{3}t^{3/2} - \frac{4}{9}t^{3/2} + c$$

d) By the method of partial fractions, $\dfrac{2}{(x+1)(x+3)} = \dfrac{1}{x+1} - \dfrac{1}{x+3}$. Thus,

$$\int_0^1 \frac{2}{(x+1)(x+3)}\, dx = \int_0^1 \frac{1}{(x+1)} - \frac{1}{x+3}\, dx = \ln\frac{x+1}{x+3}\Big|_0^1 = \ln(\frac{3}{2})$$

3. By integration by parts with $u = e^{2x}$, $du = 2e^{2x}$, and $dv = \cos x\, dx$, $v = \sin x$, and then a second integration by parts with $u = 2e^{2x}$ and $dv = \sin x\, dx$,

$$\begin{aligned}
\int_0^{\pi/2} e^{2x} \cos x\, dx &= e^{2x} \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} 2e^{2x} \sin x\, dx \\
&= e^{\pi} - \left[-2e^{2x} \cos x \Big|_0^{\pi/2} + 4 \int_0^{\pi/2} e^{2x} \cos x\, dx \right] \\
&= e^{\pi} - 2 - 4 \int_0^{\pi/2} e^{2x} \cos x\, dx
\end{aligned}$$

Solving for the integral gives $\displaystyle\int_0^{\pi/2} e^{2x} \cos x\, dx = \frac{e^{\pi} - 2}{5}$.

4. a) Use the substitution $u = 1 + x^2$, to derive $\int x/(1+x^2)^4\, dx = -(1/6)(1+x^2)^{-3} + c$. Then,

$$\begin{aligned}
\int_0^{\infty} \frac{x}{(1+x^2)^4}\, dx &= \lim_{b\to\infty} \int_0^{b} \frac{x}{(1+x^2)^4}\, dx \\
&= \lim_{b\to\infty} \left[\frac{1}{6} - \frac{1}{6(1+b^2)^3} \right] \\
&= \frac{1}{6},
\end{aligned}$$

and so the integral converges.

(b) Because $\displaystyle\frac{1}{\sqrt[3]{x^5 + x}} < \frac{1}{\sqrt[3]{x^5}} = \frac{1}{x^{5/3}}$, and because $\int_0^{\infty} 1/x^{5/3}\, dx$ converges, it follows by the comparison theorem that $\displaystyle\int_0^{\infty} \frac{1}{\sqrt[3]{x^5 + x}}\, dx$ converges.

5. (There was an error in the statement of the Midpoint rule approximation that was not important to the problem. It should read $M_4 = (1/2)\left[\ln(9/4) + \ln(11/4) + \ln(13/4) + \ln(15/4)\right]$.)

a) The Trapezoidal Rule approximation T_4 (4 subintervals) to the integral $\int_2^4 \ln x\, dx$ is

$$T_4 = \frac{1}{4}\left[\ln 2 + 2\ln(5/2) + 2\ln(3) + 2\ln(7/2) + \ln(4)\right].$$

b) The Simpson's Rule approximation S_4 (4 subintervals) to the integral $\int_2^4 \ln x\, dx$ is

$$S_4 = \frac{1}{6}\left[\ln(2) + 4\ln(5/2) + 2\ln(3) + 2\ln(7/2) + \ln(4)\right]$$

c) Let $f(x) = \ln x$. Then $f''(x) = -1/x^2$, and so $|f''(x)| = 1/x^2$. This is a decreasing function on the interval $[2, 4]$, so its maximum value is $K = |f''(2)| = (1/4)$. Thus, using the error bound formula

$$\left| T_6 - \int_2^4 \ln x\, dx \right| \le \frac{1}{4} \frac{(4-2)^3}{12(6^2)} = \frac{1}{216}.$$

6. Use the substitution $x = 2\sin\theta$, or equivalently. $\theta = \sin^{-1}(x/2)$. Then $\sqrt{4-x^2} = 2\cos\theta$ and $dx = 2\sin\theta\,d\theta$. Thus, by substitution (remember to change the limits of integration!), and then the half-angle formula,

$$\begin{aligned}\int_0^2 \sqrt{4-x^2}\,dx &= \int_{\sin^{-1}(0)}^{\sin^{-1}(1)} 4\cos^2\theta\,d\theta \\ &= \int_0^{\pi/2} 4\left(\frac{1+\cos(2\theta)}{2}\right) d\theta \\ &= [2\theta + \sin(2\theta)]\Big|_0^{\pi/2} \\ &= \pi\end{aligned}$$

7. A hemispheric tank with a radius of 2 feet is filled with water. Calculate the work required to pump out all the water through the top of the tank. (Water has a density of 62.5 lb/ft^3.)

Let x denote vertical distance in feet from the top of the tank. The horizontal cross section of the tank at distance x from the top is a circle with radius $\sqrt{4-x^2}$ ft. and hence an area of $\pi(4-x^2)$ ft^2. Therefore a slab of water in the tank at depth x and of thickness dx has a volume of $\pi(4-x^2)\,dx$ ft^2, and a weight of $62.5\pi(4-x^2)\,dx$ lbs. The work required to lift this slab of water the distance x to the top of the tank is $62.5\pi(4-x^2)x\,dx$ ft-lbs. To get the total work, the contribution for each x between 0 and 2 must be integrated:

$$W = \int_0^2 62.5\pi(4-x^2)x\,dx = 62.5\pi\left(2x^2 - \frac{x^4}{4}\right)\Big|_0^2 = 4(62.5)\pi \quad \text{ft-lbs.}$$

8. Since x^2+3x+5 is irreducible, the correct form of the partial fraction decomposition is

$$\begin{aligned}\frac{x^5+5x^3-2}{(x+2)(x-4)^2(x^2+3x+5)^2} &= \frac{A}{x+2} + \frac{B}{x-4} + \frac{C}{(x-4)^2} \\ &\quad + \frac{Dx+E}{x^2+3x+5} + \frac{Fx+G}{(x^2+3x+5)^2}.\end{aligned}$$