

Homeworks 8 and 9 Solutions, Math 154

1. (§, 11.1, problem 20)

$$\lim_{n \rightarrow \infty} \frac{n}{1 + \sqrt{n}} = \infty, \quad \text{so this sequence diverges.}$$

2. (§11.1, problem 62) A sequence is defined recursively by

$$a_1 = 2 \quad a_{n+1} = \frac{1}{3 - a_n} \quad (1)$$

a) The first problem is to show that this sequence is decreasing and bounded. To show that the sequence is decreasing, we first find the set of numbers a for which

$$a > \frac{1}{(3 - a)} \quad (2)$$

Inequality (2) is always true if $a > 3$, but this region turns out not to be important. If $a < 3$ (so that $3 - a > 0$), then (2) is true if and only if $a(3 - a) > 1$, or, equivalently, if

$$0 > 1 - 3a + a^2 = (a - r_1)(a - r_2) \quad \text{where} \quad 0 < r_1 = \frac{3 - \sqrt{5}}{2} < r_2 = 3 + \frac{\sqrt{5}}{2} < 3.$$

Hence (2) is true, for $r_1 < a < r_2$. Furthermore, if $r_1 < a < r_2$,

$$(3 - a)r_1 < (3 - r_1)r_1 = 1 \quad \text{which implies} \quad r_1 < \frac{1}{3 - a}. \quad (3)$$

Now put together inequalities (2) and (3):

$$r_1 < a < r_2 \quad \text{implies} \quad r_1 < \frac{1}{3 - a} < a < r_2. \quad (4)$$

Let $\{a_n\}$ be the solution to (1). Since $r_1 < a_1 = 2 < r_2$ and since $a_2 = 1/(3 - a_1)$, (4) implies that $r_1 < a_2 < a_1 < r_2$. But this means we can apply (4) again with $a = a_2$ and $a_3 = 1/(3 - a_2)$ to derive $r_1 < a_3 < a_2 < a_1 < r_2$. Continuing in this fashion, we see that for every n , $r_1 < a_{n+1} < a_n < r_2$. Hence $\{a_n\}$ is decreasing and bounded below by r_1 .

Since $\{a_n\}$ is decreasing and bounded below by r_1 , the Monotone Sequence Theorem implies that $\ell = \lim_{n \rightarrow \infty} a_n$ exists. Since $a_1 = 2$, $\ell < 2$. To find ℓ exactly, take limits in the recursion (1):

$$\ell = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{3 - a_n} = \frac{1}{3 - \ell}.$$

Thus ℓ must satisfy $(3 - \ell)\ell = 1$. This equation has the solutions r_1 and r_2 . But only $r_1 = (3 - \sqrt{5})/2 < 2$. Thus $\ell = r_1 = (3 - \sqrt{5})/2$.

3. (§, 11.2, problem 14)

$$1 + 0.4 + 0.16 + .064 + \dots = 1 + 0.4 + (.4)^2 + (.4)^3 + \dots = \frac{1}{1 - 0.4} = \frac{5}{3}.$$

4. (§11.2, problem 16) $\sum_{n=0}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}} = \sum_{n=0}^{\infty} \left(-\frac{6}{5}\right)^{n-1}$ diverges because $|-6/5| > 1$.

5. (§11.2, problem 38)

$$\begin{aligned} 6.2545454\dots &= 6.2 + 54(10)^{-3} + 54(10)^{-5} + 54(10)^{-7} + \dots \\ &= \frac{62}{10} + 54(10)^{-3} [1 + (10)^{-2} + (10)^{-4} + \dots] = \frac{62}{10} + 54(10)^{-3} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n \\ &= \frac{62}{10} + \frac{54}{10^3} \frac{1}{1 - (1/100)} = \frac{62}{10} + \frac{54}{990} \\ &= \frac{6192}{990} \end{aligned}$$

6. (§11.3, problem 4) To use the integral test to determine the convergence of $\sum_{n=0}^{\infty} \frac{1}{\sqrt[4]{n}}$, one must determine whether $\int_1^{\infty} \frac{1}{x^{1/4}} dx$ converges or diverges. But as this is a p -integral with $p = 1/4 \leq 1$, it diverges. Hence the infinite series diverges as well.

7. (§, 11.3, problem 28) We shall use the integral test to determine for what values of p the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges. To evaluate the integral $\int_1^{\infty} (\ln x/x^p) dx$ for $p \neq 1$, use integration by parts with $u = \ln x$ and $v = x^p$:

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x^p} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^p} dx \\ &= \lim_{b \rightarrow \infty} \left[\ln x / (1-p)x^{p-1} \Big|_1^b - \int_0^b \frac{1}{(1-p)x^p} dx \right] \\ &= \lim_{b \rightarrow \infty} \left[\ln x / (1-p)x^{p-1} - \frac{1}{(1-p)^2 x^{p-1}} \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{(1-p) \ln b - 1}{(1-p)^2 b^{p-1}} + \frac{1}{(1-p)^2} \right] \end{aligned}$$

Suppose $p > 1$. Then by L'Hopital's rule,

$$\lim_{b \rightarrow \infty} \frac{(1-p) \lim_{b \rightarrow \infty} \ln b - 1}{(1-p)^2 b^{p-1}} = \lim_{b \rightarrow \infty} \frac{b^{-1}}{(1-p)^2 b^p} = \lim_{b \rightarrow \infty} \frac{1}{(1-p)^2 b^{p-1}} = 0.$$

Therefore the integral converges for $p > 1$. On the other hand, if $p < 1$,

$$\lim_{b \rightarrow \infty} \frac{(1-p) \ln b - 1}{(1-p)^2 b^{p-1}} = \lim_{b \rightarrow \infty} \frac{1}{(1-p)^2} b^{1-p} (\ln b - 1) = \infty.$$

Thus, the integral diverges for $p < 1$. It also diverges for $p = 1$ because in this case, the integral is \int_1^∞ which diverges by the integral comparison text, because for $x > e$ $(\ln x)x > 1/x$ and we know $\int_1^\infty (1/x) dx$ converges.

By the integral test, it follows that $\sum_{n=1}^\infty \frac{\ln n}{n^p}$ converges for $p > 1$ and otherwise diverges.

8. (§, 11.3, problem 32) Find the sum of the series $s = \sum_{n=1}^\infty \frac{1}{n^5}$ correct to three decimal places. Let s_n be the partial sum of the first n terms. From formula (3) on page 718, for any positive integer n

$$s_n + \int_{n+1}^\infty \frac{1}{x^5} dx < s < s_n + \int_n^\infty \frac{1}{x^5} dx.$$

Calculation of the integral in this formula yields

$$s_n + \frac{1}{4(n+1)^4} < s < s_n + \frac{1}{4n^4}.$$

The interval $[s_n + \frac{1}{4(n+1)^4}, s_n + \frac{1}{4n^4}]$ has length $(1/4n^4) - (1/4(n+1)^4)$, so if we take the midpoint of this interval as an approximation of s , the error made is no more than

$$M_n := \frac{1}{2} \left[\frac{1}{4n^4} - \frac{1}{4(n+1)^4} \right].$$

Experiment with the calculator shows that $M_3 = .00105$ and $M_4 = .00029$. Hence, if we approximate s by the midpoint

$$\left(\frac{1}{2}\right) \left[s_4 + \frac{1}{45^4} + s_4 + \frac{1}{4(4^4)} \right] = 1.037$$

of the interval $[s_4 + \frac{1}{4(4+1)^4}, s_4 + \frac{1}{4(4^4)}]$, the approximation will be accurate to 10^{-3} .

9. (§11.4, problem 4) For each positive integer n , $\frac{2}{n^3 + 4} < \frac{2}{n^3}$. The infinite series

$\sum_{n=0}^\infty \frac{2}{n^3}$ converges because it is a p -series with $p = 3 > 1$. The comparison test therefore

implies that $\sum_{n=0}^\infty \frac{2}{n^3 + 4}$ converges as well.

10. (§11.4, problem 18) For $n > 1$, $0 < \arctan n < \pi/2$. Therefore $0 < (\arctan n)/n^4 < (\pi/2)(1/n^4)$ for all positive integers n . The infinite series $\sum_{n=0}^{\infty} \frac{\pi}{2n^4}$ converges because it is a p -series with $p = 4 > 1$. The comparison test therefore implies that $\sum_{n=0}^{\infty} \frac{\arctan n}{n^4}$ converges as well.

11. (§11.4, problem 36) Let $s = \sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$. The partial sum of the first 10 terms is

$$\sum_{n=1}^{10} \frac{n}{(n+1)3^n} = 0.283596884.$$

Because $n/(n+1) < 1$ for all positive integers n , the error $s - s_{10}$ satisfies

$$0 < s - s_{10} = \sum_{n=11}^{\infty} \frac{n}{(n+1)3^n} < \sum_{n=11}^{\infty} \frac{1}{3^n} = \frac{1}{3^{11}} \frac{1}{1 - (1/3)} = 8.5 \times 10^{-6} < 10^{-5}.$$

Thus, to state the approximation to s by s_{10} one should round off to 5 decimal places. The approximation is therefore 0.28360.