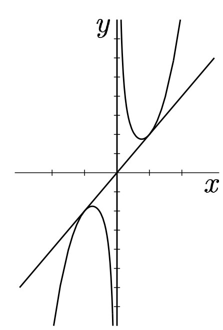
Answers, Review Hour Exam **Math 151** **Workshop 6**

1(a) Domain: $(-\infty, 4]$; Range: $[0, \infty)$. Here $f(x) = \sqrt{4-x}$ is defined if and only if $4 - x \geq 0$, or $4 \geq x$. Thus the domain is $(-\infty, 4]$. Likewise, if $y = \sqrt{4-x}$, then $y \geq 0$, and if, conversely, $y \geq 0$, the equation can be solved for x, e.g., $4 - x = y^2$, or $x = 4 - y^2$. Thus, the range is $[0, \infty)$.
1(b) Domain: $[-2, 2]$; Range: $[0, 2]$. If $y = \sqrt{4-x^2}$, then $y \geq 0$, and squaring, $y^2 = 4 - x^2$, or $x^2 + y^2 = 4$. Thus, all points on the graph of $y = f(x)$ lie on the circle with center (0,0) and radius 2. Since also $y \geq 0$, the graph is the top half of this circle. Thus, the domain and range are as indicated.

2(a) 3/5 We use $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$. In this case, specifically, replace θ by ax with a a constant. Since a is constant, $\theta = ax \to 0$ is equivalent to $x \to 0$. Thus, $\lim\limits_{x \to 0} \dfrac{\sin ax}{ax} = 1$, or $\lim\limits_{x \to 0} \dfrac{\sin ax}{x} = a$. Thus,
$$\lim_{x\to 0} \frac{\sin(3x)}{\tan(5x)} = \lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)/\cos(5x)} = \lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)} \cdot \cos(5x) = \lim_{x\to 0} \frac{\sin(3x)/x}{\sin(5x)/x} \cdot \cos(5x) = (3/5)\cdot 1 = 3/5.$$
2(b) 5/2
$$\lim_{x\to\infty} (\sqrt{x^2+5x} - x) = \lim_{x\to\infty} (\sqrt{x^2+5x} - x) \cdot \frac{(\sqrt{x^2+5x} + x)}{(\sqrt{x^2+5x} + x)} = \lim_{x\to\infty} \frac{(x^2+5x) - x^2}{\sqrt{x^2+5x} + x} =$$
$$\lim_{x\to\infty} \frac{5x}{\sqrt{x^2(1+5/x)} + x} = \lim_{x\to\infty} \frac{5x}{x\sqrt{1+5/x} + x} = \lim_{x\to\infty} \frac{5}{\sqrt{1+5/x} + 1} = 5/2.$$
2(c) 2/3
$$\lim_{x\to\infty} \frac{2x^2+3x+1}{\sqrt{9x^4+8x^3+3x+2}} = \lim_{x\to\infty} \frac{x^2(2+3/x+1/x^2)}{\sqrt{x^4(9+8/x+3/x^3+2/x^4)}} =$$
$$\lim_{x\to\infty} \frac{x^2(2+3/x+1/x^2)}{x^2\sqrt{9+8/x+3/x^3+2/x^4}} = \lim_{x\to\infty} \frac{2+3/x+1/x^2}{\sqrt{9+8/x+3/x^3+2/x^4}} = \lim_{x\to\infty} \frac{2+0+0}{\sqrt{9+0+0+0}} = \frac{2}{3}$$

3. In this problem, $f(x) = 1/\sqrt{x}$. Then, $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \left(\dfrac{1}{h}\right)\left(\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}\right) =$
$$\lim_{h\to 0} \left(\frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\right) = \lim_{h\to 0} \left(\frac{(\sqrt{x} - \sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}\right) = \lim_{h\to 0} \left(\frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}\right) =$$
$$\lim_{h\to 0} \left(\frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}\right) = \lim_{h\to 0} \left(\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}\right) = \frac{-1}{\sqrt{x}\cdot\sqrt{x}\cdot 2(\sqrt{x})} = -\frac{1}{2x^{3/2}}.$$

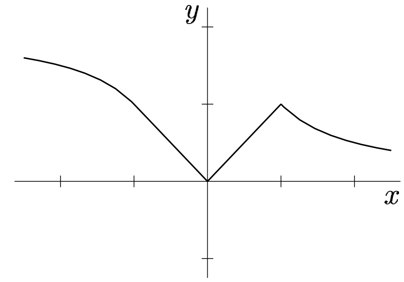
4. The graph of $y = f(x) = x^3 + \dfrac{1}{x}$ together with the tangent line at $x = 1$, whose equation is calculated in part (b) below appears at the right.



4(a) Since $y = f(x) = x^3 + \dfrac{1}{x}$, $\dfrac{dy}{dx} = f'(x) = 3x^2 - \dfrac{1}{x^2}$ If the tangent line is horizontal for a given value of x, then $\dfrac{dy}{dx} = 0$, or $3x^2 - \dfrac{1}{x^2} = 0$. Thus, $3x^4 - 1 = 0$, or $3x^4 = 1$. Thus, $x = \pm 1/3^{1/4}$, or equivalently $x = \pm 3^{-1/4}$.
4(b) At $x = 1$, $y = 2$, and $dy/dx = 3 - 1 = 2$. Thus, the equation of the tangent line is $y - 2 = 2(x-1)$, or $y = 2x$.

5(a) $\dfrac{d}{dx}(\sqrt{x^4+4x+4}) = \dfrac{d}{dx}\left((x^4+4x+4)^{1/2}\right) = (1/2)(x^4+4x+4)^{-1/2}(4x^3+4) = \dfrac{2(x^3+1)}{\sqrt{x^4+4x+4}}$.
5(b) $\dfrac{d}{dx}(x^2\sin^3(x^4)) = \dfrac{d}{dx}(x^2)\cdot\sin^3(x^4) + x^2\dfrac{d}{dx}(\sin^3(x^4)) = 2x\sin^3(x^4) + x^2(3\sin^2(x^4))\dfrac{d}{dx}(\sin(x^4)) =$
$2x\sin^3(x^4) + x^2(3\sin^2(x^4))(\cos(x^4)\cdot 4x^3) = 2x\sin^3(x^4) + 12x^5\sin^2(x^4)\cos(x^4)$.
5(c) $\dfrac{d}{dx}\left(x^2e^{-x^3}\right) = \dfrac{d}{dx}\left(x^2\right)e^{-x^3} + x^2\dfrac{d}{dx}\left(e^{-x^3}\right) = 2x\,e^{-x^3} + x^2(e^{-x^3}(-3x^2)) = (2x - 3x^4)e^{-x^3}$.
6. Let $y = f(x) = e^x/(e^x+1)$. Then, $(e^x+1)y = e^x$, or $e^xy + y = e^x$. Thus, $e^x - e^xy = e^x(1-y) = y$, or $e^x = y/(1-y)$. Taking ln( ) of both sides, $x = \ln(y/(y-1))$. Thus, $g(y) = \ln(y/(y-1))$, or $g(x) = \ln(x/(x-1))$, or $g(x) = \ln(x) - \ln(x-1)$.
7. Since $\log_u(5) = a$, $\log_u(27) = b$, $\log_u(32) = c$, $u^a = 5$, $u^b = 27$, and $u^c = 32$. Thus, it follow that:
$u^{2a+(1/3)b-(2/5)c} = u^{2a}u^{(1/3)b}u^{(-2/5)c} = (u^a)^2(u^b)^{1/3}/(u^c)^{2/5} = 5^2(27)^{1/3}/(32)^{2/5} = 5^2 3/4 = 75/4$.

8. Here, $f(1) = 2, f'(1) = 4$. Also, $g(x) = x^4 - x + 1, g'(x) = 4x^3 - 1$, and so $g(1) = 1, g'(1) = 3$. Thus,
(a) $(fg)'(1) = f(1)g'(1) + g(1)f'(1) = 2 \cdot 3 + 1 \cdot 4 = 10$
(b) $(f/g)'(1) = (g(1)f'(1) - f(1)g'(1))/g(1)^2 = (1 \cdot 4 - 2 \cdot 3)/1^2 = -2$.
(c) $(f \circ g)'(1) = f'(g(1))g'(1) = f'(1)g'(1) = 4 \cdot 3 = 12$



9(a) Since f is to be continuous, the left and right hand limits at each point must be equal. At $x = -1$, the left hand limit is $-1 + A$ and the right hand limit is 1, and so $-1 + A = 1$, or $A = 2$. Evaluating the left and right hand limits at $x = 1$ gives $1 = 1 + B$, or $B = 0$. Thus, $A = 2, B = 0$. 9(b) The graph appears to the right.
9(c) The function f fails to be differentiable at $x = 1$ and $x = 0$, since the graph has cusps at these points. More precisely, if $f'(1)$ exists, then $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0-} \frac{f(1+h) - f(1)}{h}$. Now to the left of $x = 1$, and near $x = 1$, $f(x) = |x| = x$, since $x > 0$. Thus, the left hand limit, $\lim_{h\to 0-} \frac{f(1+h) - f(1)}{h}$, is the derivative of x at $x = 1$, and so is 1. Similarly, the right hand limit, $\lim_{h\to 0+} \frac{f(1+h) - f(1)}{h}$, is the derivative at $x = 1$ of $1/x + B = 1/x$. Thus, it is $-1/x^2$, evaluated at $x = 1$, i.e., is -1. Since the left and right hand limits differ, the derivative $f'(1)$ does not exist. Similarly, at $x = 0$ the derivative $f'(0)$ does not exist.

10(a) With $f(x) = x^3 + x - 1$, then $f(0) = -1$ and $f(1) = 1$. By the intermediate value theorem, there is a point r with $0 < r < 1$, so that $f(r) = 0$, i.e., f has a root in $(0, 1)$.
10(b) Since also $g(0) = -1$ and $g(1) = 1$, g has a root in $(0, 1)$.
10(c) Here, $g(0) = -1$, $g(\frac{1}{3}) = \frac{1}{27} + \frac{1}{3} - 1 + A\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right) = \frac{-17+2A}{27}$, $g(\frac{2}{3}) = \frac{1}{27} + \frac{2}{3} - 1 + A\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{-1-2A}{27}$, $g(1) = 1$. Suppose that $A > 17/2$. Then, $g(0) < 0$, $g(\frac{1}{3}) > 0$, $g(\frac{2}{3}) < 0$, $g(1) > 0$. It follow that g has roots in each of the intervals $(0, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, 1)$, i.e., g has at least 3 roots in $[0, 1]$.

11. Since $x \to \infty$, we may assume $x > 0$. Thus, $x^2 < x^2 + x$, or $\frac{1}{x^2 + x} < \frac{1}{x^2}$. Thus, if $\frac{1}{x^2} \le \frac{1}{10^6}$, $\frac{1}{x^2 + x} < \frac{1}{x^2} \le \frac{1}{10^6}$. But $\frac{1}{x^2} \le \frac{1}{10^6}$, is equivalent to $10^6 \le x^2$, or $10^3 \le x$. Thus, we take $M = 10^3$.
Then, if $x \ge M$, $x^2 + x \ge 10^6 + 10^3 > 10^6$. Thus, $\frac{1}{x^2 + x} < \frac{1}{10^6}$.

12(a) Differentiating, $\frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(5) = 0$, or $3x^2 + 3\frac{d}{dx}(x^2)\cdot y + 3x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$. Thus, $3x^2 + 6xy + 3x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$. Solving, $\frac{dy}{dx} = -\frac{x^2 + 2xy}{x^2 + y^2}$.
12(b) Evaluating dy/dx at $(x, y) = (1, 1)$ gives the slope as $-3/2$, and from this the equation of the tangent line is $y - 1 = (-3/2)(x - 1)$, or $y = -(3/2)x + 5/2$.
12(c) $(0, 5^{1/3}), (-2, 1)$. If the tangent line is horizontal, $dy/dx = 0$, and so $x^2 + 2xy = 0$, or $x(x + 2y) = 0$. Thus, $x = 0$, or $x = -2y$. In both cases, replace x by the preceeding expression in the equation $x^3 + 3x^2y + y^3 = 5$. If $x = 0$, we obtain $y^3 = 5$, or $(x, y) = (0, 5^{1/3})$. If $x = -2y$, we obtain $-8y^3 + 12y^3 + y^3 = 5$, or $5y^3 = 5$, or $y = 1$. Since $x = -2y$, $(x, y) = (-2, 1)$.

13. In this problem, $\frac{dy}{dx} = 2x - 1$. Since (a, b) is on the parabola $y = x^2 - x$, $(a, b) = (a, a^2 - a)$. The slope of the tangent line at point (a, b) is $2a - 1$. Thus, the equation of the tangent line at (a, b) is $y - (a^2 - a) = (2a - 1)(x - a)$. Since (2,1) lies on this line, $1 - (a^2 - a) = (2a - 1)(2 - a)$, or $-a^2 + a + 1 = -2a^2 + a + 4a - 2$, or $a^2 - 4a + 3 = (a - 1)(a - 3) = 0$. Thus, $a = 1$ or $a = 3$. Thus, $(a, b) = (1, 0)$ or (3,6).

14. In the first graph, $y = f(x)$, the graph crosses the x-axis in 4 places and has a horizontal tangent line in 5 places. It follows that f has 4 roots and f' has 5 roots.
In the second graph, $y = g(x)$, the graph crosses the x-axis in 3 places and has a horizontal tangent line in 4 places. It follows that g has 3 roots and g' has 4 roots.
Then $f' \ne g$, since f' and g have a different number of roots. Thus, $g' = f$, i.e., the first graph is the graph of the derivative of the function in the second graph.