

$$1(a) \quad e^{e-1}/2. \quad \lim_{x \rightarrow e} \frac{e^x - x^e}{(x-e)^2} \stackrel{L'Hosp.}{=} \lim_{x \rightarrow e} \frac{e^x - ex^{e-1}}{2(x-e)} \stackrel{L'Hosp.}{=} \lim_{x \rightarrow e} \frac{e^x - e(e-1)x^{e-2}}{2} = \\ (e^e - e(e-1)e^{e-2})/2 = (e^e - (e-1)e^{e-1})/2 = e^{e-1}/2.$$

$$1(b) \quad 1. \quad \text{Let } y = (1+x)^{1/x}. \text{ Taking } \ln(\), \ln y = \ln((1+x)^{1/x}) = \frac{\ln(1+x)}{x}. \text{ Then, } \lim_{x \rightarrow \infty} \ln y = \\ \lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x} \stackrel{L'Hosp.}{=} \lim_{x \rightarrow \infty} \frac{1/(1+x)}{1} = 0. \text{ Thus, as } x \rightarrow \infty, \ln y \rightarrow 0. \text{ Thus, } y = e^{\ln y} \rightarrow e^0 = 1.$$

$$1(c) \quad 1. \quad \lim_{x \rightarrow \infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x} \right) = \lim_{x \rightarrow \infty} \left((\sqrt{x^2+x} - \sqrt{x^2-x}) \cdot \frac{(\sqrt{x^2+x} + \sqrt{x^2-x})}{(\sqrt{x^2+x} + \sqrt{x^2-x})} \right) = \\ \lim_{x \rightarrow \infty} \left(\frac{(x^2+x) - (x^2-x)}{\sqrt{x^2+x} + \sqrt{x^2-x}} \right) = \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2+x} + \sqrt{x^2-x}} = \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2(1+1/x)} + \sqrt{x^2(1-1/x)}} = \\ \lim_{x \rightarrow \infty} \frac{2x}{x(\sqrt{1+1/x} + \sqrt{1-1/x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1+1/x} + \sqrt{1-1/x}} = \frac{2}{1+1} = 1, \text{ as } \lim_{x \rightarrow \infty} 1/x = 0.$$

$$2(a) \quad \frac{d}{dx} \left(\frac{x^2+x+1}{x^3+x} \right) = \frac{(x^3+x)(2x+1) - (x^2+x+1)(3x^2+1)}{(x^3+x)^2} = \\ \frac{(2x^4+2x^2+x^3+x) - (3x^4+3x^3+3x^2+x^2+x+1)}{(x^3+x)^2} = \frac{-x^4-2x^3-2x^2-1}{(x^3+x)^2}.$$

$$2(b) \quad \frac{d}{dx}(x \tan(e^{-x})) = 1 \cdot \tan(e^{-x}) + x \frac{d}{dx}(\tan(e^{-x})) = \tan(e^{-x}) + x \sec^2(e^{-x}) \frac{d}{dx}(e^{-x}) = \\ \tan(e^{-x}) + x \sec^2(e^{-x})(-e^{-x}) = \tan(e^{-x}) - xe^{-x} \sec^2(e^{-x}).$$

$$2(c) \quad \text{Let } y = (1+x+x^2)^x. \text{ Taking } \ln(\), \ln y = \ln(1+x+x^2)^x = x \ln(1+x+x^2). \text{ Thus, dif-} \\ \text{ferentiating with respect to } x, \text{ it follows that } \frac{1}{y} \cdot \frac{dy}{dx} = \ln(1+x+x^2) + x \cdot \frac{d}{dx}(\ln(1+x+x^2)) = \\ \ln(1+x+x^2) + x \cdot \frac{1+2x}{1+x+x^2} = \ln(1+x+x^2) + \frac{x+2x^2}{1+x+x^2}.$$

$$\text{Thus, } \frac{dy}{dx} = \left(\ln(1+x+x^2) + \frac{x+2x^2}{1+x+x^2} \right) (1+x+x^2)^x.$$

$$2(d) \quad \sqrt{1+x^4}, \text{ by the Fundamental Theorem of Calculus.}$$

$$3(a) \quad \text{Here, } f(x) \text{ should be taken so that } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = f(0). \text{ By L'Hospital, } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \\ \lim_{x \rightarrow 0} \frac{e^x}{1} = 1. \text{ Thus, } f(0) = 1.$$

$$3(b) \quad \text{Here, } f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{((e^h - 1)/h) - 1}{h} = \lim_{h \rightarrow 0} \frac{e^h - 1 - h}{h^2} \stackrel{L'Hosp.}{=}.$$

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{2h} \stackrel{L'Hosp.}{=} \lim_{h \rightarrow 0} \frac{e^h}{2} = \frac{1}{2}. \text{ Thus, } f'(0) = 1/2.$$

$$4. \quad \text{Here, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} =$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{(2x+2h+1)^2} - \frac{1}{(2x+1)^2} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{(2x+1)^2 - (2x+2h+1)^2}{(2x+1)^2(2x+2h+1)^2} \right) =$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{(4x^2+4x+1) - (4x^2+4h^2+1+8xh+4x+4h)}{(2x+1)^2(2x+2h+1)^2} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-4h^2-8xh-4h}{(2x+1)^2(2x+2h+1)^2} \right)$$

$$= \lim_{h \rightarrow 0} \frac{-4h-8x-4}{(2x+1)^2(2x+2h+1)^2} = \frac{-8x-4}{(2x+1)^4} = \frac{-4(2x+1)}{(2x+1)^4} = \frac{-4}{(2x+1)^3}$$

5(a) Differentiating with respect to x : $3x^2 + 6xy + 3x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$, or

$$(3x^2 + 3y^2) \frac{dy}{dx} = -3x^2 - 6xy. \text{ Thus, } \frac{dy}{dx} = \frac{-3x^2 - 6xy}{3x^2 + 3y^2} = -\frac{x^2 + 2xy}{x^2 + y^2}.$$

5(b) A point on the tangent line is $(1, 1)$. To find the slope of the tangent line we evaluate the derivative at $(1, 1)$. Letting $x = 1$ and $y = 1$ in the formula for dy/dx , then $dy/dx = (-1 - 2)/2 = -3/2$. Thus, the equation of the tangent line is $y - 1 = (-3/2)(x - 1)$, or $y = (-3x + 5)/2$.

5(c) The tangent line is horizontal if $dy/dx = 0$, i.e., $x^2 + 2xy = 0$, or $x(x + 2y) = 0$. Thus, either $x = 0$ or $x = -2y$.

If $x = 0$, then from $x^3 + 3x^2y + y^3 = 5$, $y^3 = 5$, or $y = \sqrt[3]{5}$. Thus, $(x, y) = (0, \sqrt[3]{5})$.

If $x = -2y$, then from $x^3 + 3x^2y + y^3 = 5$, $(-2y)^3 + 3(-2y)^2y + y^3 = 5$, or $(-8 + 12 + 1)y^3 = 5$, or $5y^3 = 5$, or $y = 1$. Then $x = -2$.

Thus, either $(x, y) = (0, \sqrt[3]{5})$ or $(x, y) = (-2, 1)$.

6. Here, the volume, V , and the surface area, S , of a sphere of radius r are given by the formulas:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2. \text{ In addition, by hypothesis, } \frac{dV}{dt} = 5 \text{ and } \frac{dS}{dt} = 3. \text{ Differentiating the formulas}$$

with respect to t , $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi r^2 r'$, and $\frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r r'$. Thus, $5 = 4\pi r^2 r'$

and $3 = 8\pi r r'$. Dividing, $5/3 = (4\pi r^2 r')/(8\pi r r')$, or $5/3 = r/2$. Thus, $r = 10/3$ (inches).

7. By the Mean Value Theorem, $f(10) - f(0) = f'(c)(10 - 0) = 10f'(c)$, with $0 < c < 10$. As $|f'(c)| \leq 2$, $-2 \leq f'(c) \leq 2$. Thus, $-20 \leq 10f'(c) \leq 20$. It follows that $-20 \leq f(10) - f(0) \leq 20$, or $-20 + f(0) \leq f(10) \leq 20 + f(0)$. Since $f(0) = 100$, $80 \leq f(10) \leq 120$. Thus, $A = 80$, $B = 120$.

8. At $x = -5$ and $x = 3$, $f'(x)$ changes sign from positive to negative, and thus a local maximum occurs at $x = -5$ and $x = 3$.

At $x = 0$ and $x = 6$, $f'(x)$ changes sign from negative to positive, and thus a local minimum occurs at $x = 0$ and $x = 6$.

At $x = -3$ and $x = 1$, $f'(x)$ does not change sign, and neither occurs.

9(a) Here, $y = f(x) = \frac{1}{x} + \ln x$. The domain is $(0, \infty)$, since $\ln x$ is only defined if $x > 0$.

As $x \rightarrow 0^+$, $y \rightarrow \infty - \infty$, and to evaluate the limit, we collect the terms together. Then, $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(\frac{1 + x \ln x}{x} \right)$. Observe that $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \left(\frac{\ln x}{1/x} \right)$. As this is $\frac{-\infty}{\infty}$, we

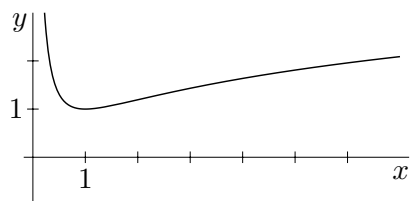
use L'Hospital's rule to get: $\lim_{x \rightarrow 0^+} \left(\frac{1/x}{-1/x^2} \right) = \lim_{x \rightarrow 0^+} -x = 0$. Thus, as $x \rightarrow 0^+$, $x \ln x \rightarrow 0$. Then,

as $x \rightarrow 0^+$, $1 + x \ln x \rightarrow 1 + 0 = 1$. It follows that $\lim_{x \rightarrow 0^+} \left(\frac{1 + x \ln x}{x} \right) = +\infty$. Thus, the graph has a vertical asymptote at $x = 0$.

Since $\lim_{x \rightarrow \infty} \left(\frac{1}{x} + \ln x \right) = \infty$, the graph has no horizontal asymptote.

Differentiating, $\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{x} = \frac{-1 + x}{x^2}$. Thus, if $dy/dx = 0$, $x = 1$, and the only critical point is $x = 1$. Also, $\frac{d^2y}{dx^2} = \frac{2}{x^3} - \frac{1}{x^2} = \frac{2 - x}{x^3}$. Thus, if $d^2y/dx^2 = 0$, $x = 2$, and the only inflection point is $x = 2$.

Since $\lim_{x \rightarrow 0^+} f(x) = \infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$, the function has a minimum at $x = 1$. Thus, as $f(1) = 1$, the range of the function is $[1, \infty)$. The graph of the function appears to the right.

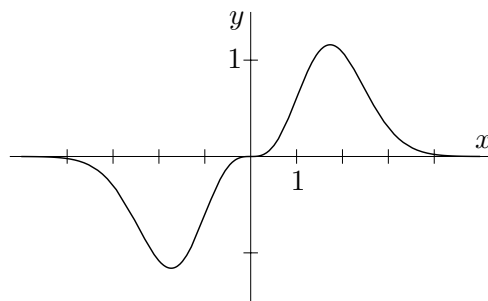


9(b) For this function, $dy/dx = 3x^2e^{-x^2/2} + x^3(-x)e^{-x^2/2} = (3x^2 - x^4)e^{-x^2/2}$. Thus, if $dy/dx = 0$, then $x^4 - 3x^2 = x^2(x^2 - 3) = 0$. It follows that $x = 0$ or $x = \pm\sqrt{3}$. Thus, the critical points of the function are $x = 0$ and $x = \pm\sqrt{3}$.

Similarly, $d^2y/dx^2 = (6x - 4x^3)e^{-x^2/2} + (3x^2 - x^4)(-x)e^{-x^2/2} = (x^5 - 7x^3 + 6x)e^{-x^2/2}$. Thus to be an inflection point, x must satisfy the equation $x^5 - 7x^3 + 6x = 0$, or $x(x^2 - 1)(x^2 - 6) = 0$. Thus, the inflection points are $x = 0$, $x = \pm 1$, and $x = \pm\sqrt{6}$.

Since $\lim_{x \rightarrow \infty} x^3 e^{-x^2/2} = \lim_{x \rightarrow \infty} \frac{x^3}{e^{x^2/2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{-xe^{x^2/2}} = \lim_{x \rightarrow \infty} \frac{-3x}{e^{x^2/2}} = \lim_{x \rightarrow \infty} \frac{-3}{-xe^{x^2/2}} = 0$, the horizontal asymptote of the graph is the line $y = 0$.

From the graph, the maximum value of the function occurs at $x = \sqrt{3}$ and is $M = 3^{3/2}e^{-3/2} = (3/e)^{3/2}$. Likewise, the minimum is $-3^{3/2}e^{-3/2} = -(3/e)^{3/2}$. Thus, the range is the interval $[-(3/e)^{3/2}, (3/e)^{3/2}] = [-1.159418, 1.159418]$.



10(a) Since the linear approximation at $x = a$ of f is $L(x) = f(a) + f'(a)(x - a)$, with $a = 1$, $f(1) = 3$, $f'(1) = -2$, $L(x) = 3 - 2(x - 1)$.

10(b) Setting $x = 1.05$, $L(1.05) = 3 - 2(1.05 - 1) = 2.9$

10(c) Here $g(x) = x^2 f(x)$. Then $g(1) = 1^2 f(1) = 3$. Also $g'(x) = 2x f(x) + x^2 f'(x)$. Setting $x = 1$, $g'(1) = 2f(1) + f'(1) = 2 \cdot 3 - 2 = 4$. Thus, denoting the linear approximation of $g(x)$ at $a = 1$ by $M(x)$, then $M(x) = 3 + 4(x - 1)$.

$$11(a) \int \frac{x^2 + x + 1}{\sqrt{x}} dx = \int (x^{3/2} + x^{1/2} + x^{-1/2}) dx = (2/5)x^{5/2} + (2/3)x^{3/2} + 2x^{1/2} + C.$$

$$11(b) \text{ With } u = x + 1, du = dx. \text{ Thus, } \int \sqrt{x+1} dx = \int \sqrt{u} du = (2/3)u^{3/2} + C = (2/3)(x+1)^{3/2} + C. \text{ Thus, } \int_0^1 \sqrt{x+1} dx = (2/3)(x+1)^{3/2} \Big|_0^1 = (2/3)(2^{3/2} - 1).$$

11(c) Take $u = x^3$. Then, $du = 3x^2 dx$, or $x^2 dx = (1/3)du$.

$$\text{Then, } \int x^2 \sin(x^3) dx = \int (1/3) \sin u du = -(1/3) \cos u + C = -(1/3) \cos(x^3) + C.$$

11(d) First perform the integral $\int e^{ax} dx$, with a a non-zero constant. Letting $u = ax$, then $du = a dx$ and $dx = (1/a)du$. Thus, $\int e^{ax} dx = \int e^u (1/a) du = (1/a)e^u + C = (1/a)e^{ax} + C$.

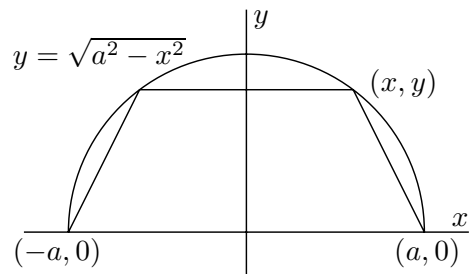
$$\text{Thus, } \int_{-1}^1 (e^{2x} + e^{-x}) dx = ((1/2)e^{2x} - e^{-x}) \Big|_{-1}^1 = ((1/2)e^2 - e^{-1}) - ((1/2)e^{-2} - e^1) = (1/2)(e^2 - e^{-2}) + e - e^{-1}.$$

12(a) Here, $a(t) = \frac{d^2x}{dt^2} = -24t^2 + 96t$. Thus, $v(t) = \frac{dx}{dt} = \int (-24t^2 + 96t) dt = -8t^3 + 48t^2 + C$, where C is a constant. Since $v(0) = 0$, $0 = -8 \cdot 0^3 + 48 \cdot 0^2 + C$, and $C = 0$.

Thus, $v(t) = \frac{dx}{dt} = -8t^3 + 48t^2$, and so, $x(t) = \int (-8t^3 + 48t^2) dt = -2t^4 + 16t^3 + D$, with D a constant. Since $x(0) = 0$, $0 = 0 + 0 + D$, or $D = 0$. It follows that $x(t) = -2t^4 + 16t^3$.

12(b) The particle goes down the x -axis until $dx/dt = 0$. Since $dx/dt = -8t^3 + 48t^2$, $dx/dt = 0$ if $t = 0$ or $t = 6$. At $t = 0$, $x = 0$, and at $t = 6$, $x = 864$. Thus, $x = 864$ is as far as the particle goes down the positive x -axis.

13. Let (x, y) be the upper right hand corner of the trapezoid. Then, the top side of the trapezoid has width $2x$, the bottom side has width $2a$, and the height is y . Thus, if A is the area $A = (2x+2a)y/2 = (x+a)y$. Since (x, y) is a point on the semi-circle, $y = \sqrt{a^2 - x^2}$ and $A = (x+a)\sqrt{a^2 - x^2}$



Then, $\frac{dA}{dx} = \sqrt{a^2 - x^2} + (x+a) \left(\frac{-x}{\sqrt{a^2 - x^2}} \right) = \frac{(a^2 - x^2) - x(x+a)}{\sqrt{a^2 - x^2}} = \frac{-2x^2 - ax + a^2}{\sqrt{a^2 - x^2}} = \frac{-2(x^2 + (a/2)x - a^2/2)}{\sqrt{a^2 - x^2}} = \frac{-2(x+a)(x-a/2)}{\sqrt{a^2 - x^2}}$. Thus, the critical points of A are $x = -a$ and $x = a/2$. Since the problem only makes sense for $0 \leq x \leq a$, the relevant critical point is $x = a/2$.

At the endpoint $x = 0$, $A(0) = a^2$. At the critical point $x = a/2$, $A(a/2) = 3\sqrt{3}a^2/4$. Finally, at the endpoint $x = a$, $A(a) = 0$. Thus, the maximum value is $A = 3\sqrt{3}a^2/4$, and it occurs at $x = a/2$.

14(a) The area of a semi-circle of radius 1 is $\pi/2$. The trapezoidal regions below $[2, 5]$ and $[7, 10]$ each have total area 2.

(i) $\pi/2$ (ii) $\pi/2 - 2$ (iii) $\pi - 2$ (iv) $\pi - 4$ (v) $\pi/2 - 3$

14(b) By the Fundamental Theorem of Calculus, $F'(x) = f(x)$. From the graph $f(x) = 0$ at $x = 0, 2, 5, 7, 10$, and, consequently, these are the critical points of F . In this case, the endpoints of the interval $[0, 10]$ happen to be critical points. Thus, the maximum value of F on the interval $[0, 10]$ is the largest of the numbers $F(0), F(2), F(5), F(7), F(10)$, and the minimum value of F is the smallest of these numbers.

Then, $F(0) = 0$, and by part (a), $F(2) = \pi/2$, $F(5) = \pi/2 - 2$, $F(7) = \pi - 2$, $F(10) = \pi - 4$. Thus, the maximum value of F on the interval $[0, 10]$ is $\pi/2$, taken at $x = 2$, and the minimum value is $\pi - 4$, taken at $x = 10$.

14(c) Here, $\int_6^1 f(x) dx = -\int_1^6 f(x) dx$. Also, $\int_1^6 f(x) dx = \pi/4 - 2 + \pi/4 = \pi/2 - 2$.

Thus, $\int_6^1 f(x) dx = 2 - \pi/2$.

15. Since $[1, 3]$ is divided into four subintervals of equal length, the width of each subinterval is $\Delta x = (3 - 1)/4 = 1/2$. Writing 1 as $2/2$ and 3 as $6/2$, the subintervals are:

$[2/2, 3/2]$, $[3/2, 4/2]$, $[4/2, 5/2]$, $[5/2, 6/2]$. Here, also $f(x) = \frac{1}{x}$ and $\Delta x = 1/2$.

15(a) The left endpoints of the subintervals are: $2/2, 3/2, 4/2, 5/2$. Thus, the Riemann sum is: $(1/2)(f(2/2) + f(3/2) + f(4/2) + f(5/2)) = \frac{1}{2} \left(\frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} \right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$.

15(b) The right endpoints of the subintervals are: $3/2, 4/2, 5/2, 6/2$. Thus, the Riemann sum is: $(1/2)(f(3/2) + f(4/2) + f(5/2) + f(6/2)) = \frac{1}{2} \left(\frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} \right) = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{57}{60}$.

15(c) The midpoints of the subintervals are: $5/4, 7/4, 9/4, 11/4$. Thus, the Riemann sum is: $(1/2)(f(5/4) + f(7/4) + f(9/4) + f(11/4)) = \frac{1}{2} \left(\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} \right) = 2 \left(\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \right) = \frac{3776}{3465}$.