Solutions to the second exam

1. a) Let $a_n = ne^{-n}$ for integers $n \geq 1$. Does the sequence $\{a_n\}$ converge? If so, find its limit.

By L'Hopital's rule, $\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$. Hence the sequence converges and its limit is 0.

b) Is the infinite series $\sum_1^\infty ne^{-n}$ convergent or divergent? Justify your answer using a convergence test.

The function $f(x) = xe^{-x}$ is decreasing for $x > 1$, and $\int_1^\infty xe^{-x}\,dx = -(x+1)e^{-x}\,\Big|_1^\infty$ $= 2/e$ converges. It follows by the integral test that $\sum_1^\infty ne^{-n}$ converges.

Notes: This problem was written in part to test your understanding of the difference between convergence of a sequence and convergence of a series. Make sure you understand the distinction!

Par b) can also be done easily using the ratio rule.

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2. Find the equation for the curve passing through the point $x = 0$, $y = 1/2$ and whose slope at any point (x, y) is xy^2.

This problem translates to: find the solution of

$$\frac{dy}{dx} = xy^2 \qquad y(0) = 2$$

By separation of variables, $\int y^{-2}\,dy = \int x\,dx$. Integration gives $-y^{-1} = (x^2/2) + c$, and the initial condition implies $-1/(1/2) = 0 + c$, that is, $c = -2$. Hence $-y^{-1} = (x^2/2) - 2$, and, solving for y,

$$y = \frac{1}{2 - (x^2/2)} = \frac{2}{4 - x^2}.$$

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4. In each case determine whether the series diverges, converges conditionally, or converges absolutely. In each case, state either the test or the formula you use and show how it applies.

a) $\sum_1^\infty \frac{3 + \sin n}{n^{3/2}}$ **converges absolutely.** The reason is that for every positive integer n $\left|\frac{3+\sin n}{n^{3/2}}\right| \leq \frac{4}{n^{3/2}}$ and $\sum_1^\infty \frac{4}{n^{3/2}}$ converges because it is a p-series with $p = 3/2 > 1$. By the comparison test $\sum_1^\infty \left|\frac{3+\sin n}{n^{3/2}}\right|$ converges.

b) $\sum_1^\infty (-1)^n \frac{2n}{5n+2}$ diverges by the divergence test, since $\lim_{n\to\infty}(-1)^n(2n/(5n+2)$ does not exist. (Note that $\lim_{n\to\infty}(-1)^n(2n/(5n+2) = 2/5 \neq 0.)$

c) $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^{1/3}}$ is conditionally convergent. It converges by the alternating series test. However, $\sum_{1}^{\infty} \left| \frac{(-1)^{n+1}}{(n+1)^{1/3}} \right| = \sum_{1}^{\infty} \frac{1}{(n+1)^{1/3}}$ diverges as it is a p-series with $p = 1/3$.

d) $\sum_{1}^{\infty} \frac{2^n}{(n+1)!}$ converges by the ratio test, since, with $a_n = 2^n/(n+1)!$, it follows that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2}{n+2} = 0 < 1$.

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5. Consider the curve represented by the graph of the function $y = (3/2)(x^2 - 2)^{3/2}$ for $1 \le x \le 2$. Represent the length of this curve as an integral. The answer is $L = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \sqrt{1 + (81/4)x^2(x^2 - 2)}\, dx$.

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6. The arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ is rotated about the y-axis. Find the surface area of the surface of revolution. Since the curve is revolved about the y-axis,

$$\text{surface area} = \int_0^1 2\pi x(y)\sqrt{1 + (dx/dy)^2}\, dy = \int_0^1 2\pi\sqrt{y}\sqrt{1 + (1/2\sqrt{y})^2}\, dy$$

$$= \int_0^1 2\pi\sqrt{y + (1/4)}\, dy = \frac{4\pi}{3}(y + (1/4))^{3/2} \Big|_1^2 = \frac{\pi}{6}(5^{3/2} - 1).$$

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7. Calculate the sum or show that the series diverges.

a) $\sum_{n=1}^{\infty} \frac{1+5^{n+1}}{8^n} = \frac{1}{8}\sum_{n=1}^{\infty}(\frac{1}{8})^{n-1} + \frac{25}{8}\sum_{n=1}^{\infty}(\frac{5}{8})^{n-1} = \frac{1}{8}\frac{1}{1-(1/8)} + \frac{25}{8}\frac{1}{1-(5/8)} = \frac{1}{7} + \frac{25}{3}$

b) $\sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+2} = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \ldots + \frac{1}{N} - \frac{1}{N+2} = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$. Therefore

$$\sum_{1}^{\infty} \frac{1}{n} - \frac{1}{n+2} = \lim_{N\to\infty} 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} = \frac{3}{2}.$$

c) $\sum_{0}^{\infty} 2\left(-\frac{5}{4}\right)^n$ diverges, as this is a geometric series with $r = -5/4$ and $|r| = 5/4 > 1$.

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a) Engineer A approximates $s = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^{5/2}}$ using the sum

$$1 - \frac{1}{2^{5/2}} + \frac{1}{3^{5/2}} - \frac{1}{4^{5/2}} + \frac{1}{5^{5/2}} - \frac{1}{6^{5/2}} + \frac{1}{7^{5/2}} - \frac{1}{8^{5/2}}$$

This is the partial sum s_8, and, since the series is alternating, $|s - s_8| \le b_9 = (1/9^{5/2}) = 1/243$. s_8 underestimates s because, since the next term in the sum is positive and $b_n = 1/n^{5/2}$ is decreasing, $s_8 < s < s_9$.

b) Engineer B wants to approximate $\sum_{1}^{\infty} \frac{1}{n^{5/2}}$ with a partial sum $s_N = \sum_{1}^{N} \frac{1}{n^{5/2}}$. How large should he take N so that the error is less than 0.005 ?

Since $0 < s - s_N < \int_N^{\infty}(1/x^{5/2})\, dx = -\frac{2}{3}x^{-3/2} \Big|_N^{\infty} = \frac{2}{3N^{3/2}}$, he should choose N so that $\frac{2}{3N^{3/2}} \le \frac{5}{1000}$. Thus, engineer B should take $N \ge (400/3)^{2/3}$.