

Homeworks 10 Solutions, Math 154

1. (§, 11.5, problem 8) The sequence $\left\{\frac{2n}{4n^2 + 1}\right\}$ is decreasing and $\lim_{n \rightarrow \infty} 2n/(4n^2 + 1) = 0$. The limit as $n \rightarrow \infty$ is easy to see. To see that the sequence is decreasing, consider

$$f(x) = \frac{2x}{4x^2 + 1} = \frac{2}{4x + (1/x)}.$$

The derivative of the denominator $4x + 1/x$ is $4 - (1/x^2)$, and this is positive for all $x \geq 1$. Thus the denominator is increasing on the interval $x > 1$, and so $f(x)$ is decreasing on the interval $x > 1$. This proves that the sequence $\left\{\frac{2n}{4n^2 + 1}\right\}$ is decreasing. These arguments show that the alternating series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2 + 1}$$

satisfies the conditions of the alternating series convergence theorem. Hence it converges.

2. (§11.5, problem 22) Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$. Let s_{10} be the partial sum of the first 10 terms. According to the Alternating Series Estimation Theorem,

$$|s - s_{10}| \leq \frac{1}{n^3} \Big|_{n=10} = 10^{-3}.$$

3. (§, 11.6, problem 4) The infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ diverges by the alternating series theorem, but it is not absolutely convergent because $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p -series with $p = 1/2$, and hence diverges. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent.

4. (§, 11.6, problem 6) The infinite series $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ is absolutely convergent by the ratio test, because

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0.$$

5. (§11.6, problem 12) The infinite series $\sum_{n=1}^{\infty} e^{-n} n!$ is divergent by the ratio test since

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} e^{-1}(n+1) = \infty.$$

6. (§11.7, problem 14) The infinite series $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$ diverges. To see this, first note that

$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$ by the integral test:

$$\int_1^{\infty} \frac{x^2}{x^3 + 1} dx = \lim_{b \rightarrow \infty} (1/3) \ln(x^3 + 1) \Big|_1^b = \infty.$$

But for every positive integer n , $\frac{n^2 + 1}{n^3 + 1} > \frac{n^2}{n^3 + 1}$. Hence by the comparison test,

$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1} \quad \text{diverges as well.}$$

This problem could easily be done by the limit comparison test, instead of the approach taken here.