Solutions to selected review problems

These solutions focus on problems on the review sheet that we did not discuss adequately in class.

2. (b) When x is large $x^{2/3}$ is the dominant term in the numerator of the integrand. Therefore we expect the integral to diverge because $\int_1^\infty \frac{1}{x^{2/3}}\,dx$ diverges. To show this using the comparison test we want to find K so that

$$\frac{K}{x^{2/3}} \le \frac{1}{x^{1/2}+x^{2/3}}, \quad \text{for all } x \ge 1.$$

To do this observe that $x^{2/3}+x^{1/3} = x^{2/3}[1+x^{-1/3}]$. Since $1+x^{-1/3}$ is a decreasing function, its maximum on $1 \le x < \infty$ is $1+1^{-1/3}=2$. Therefore $x^{2/3}+x^{1/3} \le 2x^{2/3}$ for all $x \ge 1$, and it follows that

$$\frac{1}{2}\frac{1}{x^{2/3}} \le \frac{1}{x^{1/2}+x^{2/3}}, \quad \text{for all } x \ge 1.$$

Conclusion: because of this inequality and the fact that $\int_1^\infty \frac{1}{2}\frac{1}{x^{2/3}}\,dx$ diverges, the integral comparison test implies $\int_1^\infty \frac{1}{x^{1/3}+x^{2/3}}\,dx$ diverges.

4. If $r = 5\sin\theta$, then $r^2 = 5r\sin\theta$. Using $r^2 = x^2+y^2$ and $y = r\sin\theta$, it follows that

$$x^2+y^2 = 5y \quad \text{or, equivalently} \quad x^2+y^2-5y = 0.$$

By completion of squares, $y^2-5y = (y-(5/2))^2-(25/4)$. Thus we obtain

$$x^2+(y-(5/2))^2 = 25/4,$$

which is the equation of a circle centered at $(0,5/2)$ of radius $5/2$.

6. See problem 5, section 11.4. The cardoid is swept out by the equation $r(\theta) = 1-\cos\theta$ as θ varies from 0 to 2π. Hence the area of the cardoid is

$$\frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}[1-2\cos\theta+\cos^2\theta]\,d\theta = \frac{1}{2}\int_0^{2\pi}[1-2\cos\theta+\frac{1+\cos 2\theta}{2}]\,d\theta.$$

The integrals of the cosine terms are zero. Hence the answer is $3\pi/2$.

7. The first four derivatives of $f(x) = (1+x)^{5/4}$ are $f'(x) = (5/4)(1+x)^{1/4}$, $f''(x) = (5/16)(1+x)^{-3/4}$, $f^{(3)} = -(15/64)(1+x)^{-7/4}$, $f^{(4)}(x) = (105/256)(1+x)^{-11/4}$.

Thus $f(0) = 1$, $f'(0) = 5/4$, $f''(0) = 5/16$, $f^{(3)}(0) = -15/64$, and so

$$T_3(x) = 1 + \frac{5}{4}x + \frac{5}{32}x^2 - \frac{15}{64 \cdot 6}x^3.$$

By the remainder formula $f(-0.1) - T_3(-0.1)$ equals

$$\frac{1}{3!}\int_0^{-.1}(-.1-u)^3 f^{(4)}(u)\,du = -\frac{1}{3!}\int_0^{-.1}(.1+u)^3 f^{(4)}(u)\,du = \frac{1}{3!}\int_{-.1}^{0}(.1+u)^3 f^{(4)}(u)\,du$$

All terms in the integrand of the last expression are positive and hence $f(-0.1) - T_3(0.1) > 0$.

Since $f^{(4)}(x)$ is a decreasing function its maximum value over $-0.2 \le x \le 0.2$ is $f^{(4)}(-.2) = (105/256) \cdot (.8)^{-11/4}$. Thus if $-0.2 \le x \le 0.2$,

$$\left| f(x) - T_3(x) \right| \le \frac{1}{4!}\frac{105}{256(.8)^{11/4}}|x|^4 \le \frac{1}{4!}\frac{105(.2)^4}{256(.8)^{11/4}}.$$

Chapter 10 review, 30.

$$\sum_{n=2}\left(\frac{2}{e}\right)^n = \left(\frac{2}{e}\right)^2 + \left(\frac{2}{e}\right)^3 + \cdots = \left(\frac{2}{e}\right)^2\left[1 + \left(\frac{2}{e}\right) + \left(\frac{2}{e}\right)^2 + \cdots\right] = \left(\frac{2}{e}\right)^2 \frac{1}{1-(2/e)}.$$

Chapter 10 review, 34. The n^{th} circle from the left has radius 2^{-n-1} and therefore has area $\pi 2^{-2n-2} = \pi 4^{-n} 4^{-1}$. Thus the total area of all the circles is

$$\sum_{n=1}^{\infty}\frac{\pi}{4}\left(\frac{1}{4}\right)^n = \frac{\pi}{16}\frac{1}{1-\frac{1}{4}} = \frac{\pi}{12}.$$

(Notice that the sum starts at $n = 1$.)

Chapter 10 review, 36. Observe that $\dfrac{n^2}{(n^3+1)^{1.01}} < \dfrac{n^2}{(n^3)^{1.01}} = \dfrac{1}{n^{1.03}}$. Since $\displaystyle\sum_1^{\infty}\frac{1}{n^{1.03}}$ converges (it is a p-series with $p = 1.03 > 1$), it follows from the comparison test that $\displaystyle\sum_1^{\infty}\frac{1}{n^{1.03}}$ converges also.

Chapter 10 review, 44. Because $\dfrac{n}{\sqrt{n^5+5}} < \dfrac{n}{n^{5/2}} = \dfrac{1}{n^{3/2}}$ and because $\displaystyle\sum_1^{\infty}\frac{1}{n^{3/2}}$ converges, the comparison test implies $\displaystyle\sum_1^{\infty}\frac{n}{\sqrt{n^5+5}}$ converges also.

It can be shown that $\dfrac{x}{\sqrt{x^5+5}}$ is decreasing on $2 \le x < \infty$. Thus

$$0 < \sum_1^\infty \frac{n}{\sqrt{n^5+5}} - \sum_1^N \frac{n}{\sqrt{n^5+5}} \le \int_N^\infty \frac{x}{\sqrt{x^5+5}}\,dx \le \int_N^\infty \frac{1}{x^{3/2}}\,dx = \frac{2}{\sqrt{N}}.$$

Thus the partial sum $\displaystyle\sum_1^N \frac{n}{\sqrt{n^5+5}}$ approximates the infinite series to an accuracy of 10^{-3} if $2/\sqrt{N} < 10^{-3}$ or $N > 4 \times 10^6$.

Chapter 10 review, 30. Observe that

$$\lim_{n\to\infty} \frac{1/(3^n-2^n)}{1/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n-2^n} = \lim_{n\to\infty} \frac{1}{1-(2/3)^n} = 1.$$

Since this limit is finite and since $\sum_1^\infty \frac{1}{3^n}$ converges, because it is a geometric series with $|r| = 1/3 < 1$, it follows from the limit comparison test that $\displaystyle\sum_1^\infty \frac{1}{3^n-2^n}$ converges also.

Chapter 10 review, 51. This series converges by the Leibniz test for alternating series (see page 585). But it converges absolutely as well by the comparison test, because

$$\left| \frac{(-1)^n}{n^{1.1}\ln(n+1)} \right| \le \frac{1}{\ln 2}\frac{1}{n^{1.1}}, \qquad \text{for } n \ge 1,$$

and $\displaystyle\sum_1^\infty \frac{1}{n^{1.1}}$ converges.

Chapter 10 review, 61. This converges by the ratio test, since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^4}{n^4}\frac{1}{n+1} = 0 < 1.$$

Chapter 10 review, 71. This converges because it is a geometric series with $|r| = 2/3 < 1$.

Chapter 10 review, 73. This is a geometric series with $|r| = e^{-0.02} < 1$ and hence it converges.

Chapter 10 review, 75.. This converges conditionally but not absolutely. It converges by Leibniz's alternating series test. But $\dfrac{1}{\sqrt{n}+\sqrt{n+1}} > \dfrac{1}{2\sqrt{n+1}}$, and $\displaystyle\sum_1^\infty \frac{1}{\sqrt{n+1}}$, diverges. Hence so does $\displaystyle\sum_1^\infty \frac{1}{\sqrt{n}+\sqrt{n+1}}$ and the alternating series is not absolutely convergent.