

Review problem solutions

1. (a)  $S = \int_0^{1/2} \sqrt{1 + (dy/dx)^2} dx = \int_0^{1/2} \sqrt{1 + x^6} dx.$

(b)  $\sqrt{1 + x^6} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{6n} = 1 + \frac{x^6}{2} + \frac{(1/2)(1/2-1)}{2!} x^{12} + \frac{(1/2)(-1/2)(-3/2)}{3!} x^{18} + \dots$   
 $= 1 + \frac{1}{2}x^6 - \frac{1}{8}x^{12} + \frac{1}{16}x^{18} + \dots$  for  $|x| < 1.$

Hence,  $\int_0^{1/2} \sqrt{1 + x^6} = \sum_{n=0}^{\infty} \binom{1/2}{n} \int_0^{1/2} x^{6n} dx = \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{(1/2)^{6n+1}}{6n+1}$   
 $= \frac{1}{2} + \frac{1}{2} \frac{(1/2)^7}{7} - \frac{1}{8} \frac{(1/2)^{13}}{13} + \dots$

(c) The method of (b) does not work on the interval  $0 \leq x \leq 2$  because the Maclaurin series of  $\sqrt{1 + x^6}$  converges only for  $|x| < 1$ . The Trapezoid method with  $n = 4$  gives

$$T_4 = \frac{1}{16} \left[ \sqrt{1} + 2\sqrt{1 + (1/8)^6} + 2\sqrt{1 + (1/4)^6} + 2\sqrt{1 + (3/8)^6} + \sqrt{1 + (1/2)^6} \right]$$

(d) The error of  $T_4$  is less than  $\frac{13(1/2)^3}{12 \cdot 4^2}.$

2. Using partial fraction decomposition,

$$\frac{x^2 - 4x - 9}{x^2 - 1} = 1 - \frac{4x + 8}{x^2 - 1} = 1 - \frac{6}{x - 1} + \frac{2}{x + 1}.$$

Thus,  $\int \frac{x^2 - 4x - 9}{x^2 - 1} dx = x - 6 \ln|x - 1| + 2 \ln|x + 1| + c.$  By separation of variables,  $\ln|y| = x - 6 \ln|x - 1| + 2 \ln|x + 1| + c$  and using the initial condition  $y(2) = 3$ ,  $\ln 3 = 3 + 2 \ln 3 + c$  so  $c = -3 - \ln 3$ . By exponentiating

$$|y| = \exp\{x - 6 \ln|x - 1| + 2 \ln|x + 1| - 3 - \ln 3\}.$$

Since  $y(2) > 0$ , we take away the absolute value:  $y = \frac{e^{-3}}{3} (x + 1)^2 (x - 1)^{-6} e^x.$

3. The area of  $R$  is  $\int_{-\infty}^0 e^x dx = \lim_{b \rightarrow \infty} \int_{-b}^0 e^x dx = \lim_{b \rightarrow \infty} 1 - e^{-b} = 1.$

(b) The volume when  $R$  is revolved about the  $x$ -axis is, by the disk method, the substitution  $u = 2x$ , and the calculation of part (a),

$$V = \pi \int_{-\infty}^0 e^{2x} dx = \pi \frac{1}{2} \int_{-\infty}^0 e^u du = \frac{\pi}{2}.$$

For the volume about the  $y$ -axis, we use the shell method and then integration by parts,

$$V = \int_{-\infty}^0 2\pi|x|e^x dx = - \int_{-\infty}^0 2\pi x e^x dx = -2\pi x e^x \Big|_{-\infty}^0 + 2\pi \int_{-\infty}^0 e^x dx = 2\pi$$

4. (a) Write the integral as  $\int \frac{x}{\sqrt{4+x^2}} dx + \int \frac{1}{\sqrt{4+x^2}} dx$ . The first integral we do by the substitution  $u = 4 + x^2$  the second by the substitution  $\tan \theta = x/2$ . Thus, using an integration formula from the formula sheet,

$$\begin{aligned} \int \frac{x}{\sqrt{4+x^2}} dx &= \frac{1}{2} \int \frac{du}{\sqrt{u}} = \sqrt{u} + c = \sqrt{4+x^2} + c \\ \int \frac{1}{\sqrt{4+x^2}} dx &= \int \left(\frac{1}{2} \cos \theta\right) 2 \sec^2 \theta d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + c \\ &= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + c \end{aligned}$$

(Note:  $\ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + c = \ln |\sqrt{4+x^2} + x| - \ln 2 + c = \ln |\sqrt{4+x^2} + x| + c'$ .)

The final answer is  $\sqrt{4+x^2} + \ln |\sqrt{4+x^2} + x| + c$ .

(b) Use  $u = \sqrt{x}$ , which leads to  $dx = 2u du$  and, with integration by parts,

$$\begin{aligned} \int \sqrt{x} \sin \sqrt{x} dx &= \int 2u^2 \sin u du = -2u^2 \cos u + 4 \int u \cos u du \\ &= -2u^2 \cos u + 4u \sin u - \int 4 \sin u du \\ &= -2u^2 \cos u + 4u \sin u + 4 \cos u + c \\ &= -2x \cos \sqrt{x} + 4\sqrt{x} \sin \sqrt{x} + 4 \cos \sqrt{x} + c. \end{aligned}$$

(c) Observe that  $5 - 4x - x^2 = 9 - (x + 2)^2$  and use the trigonometric substitution  $x + 2 = 3 \sin \theta$ . Then  $dx = 3 \cos \theta$  and  $\sqrt{9 - (x + 2)^2} = 3 \cos \theta$ . So, using the half-angle formula, and  $\sin 2\theta = 2 \sin \theta \cos \theta$ ,

$$\begin{aligned} \int \sqrt{5 - 4x - x^2} dx &= 9 \int \cos^2 \theta d\theta = \frac{9}{2} \theta + \frac{9}{4} \sin(2\theta) + c = \frac{9}{2} [\theta + \sin \theta \cos \theta] + c \\ &= \frac{9}{2} \left[ \sin^{-1}[(x+2)/3] + \frac{(x+2)\sqrt{9-(x+2)^2}}{9} \right] + c \end{aligned}$$

$$\begin{aligned} 5. \int_0^{\pi/4} \tan^4 \theta d\theta &= \int_0^{\pi/4} \tan^2 \theta [\sec^2 \theta - 1] d\theta = \frac{\tan^3 \theta}{3} \Big|_0^{\pi/4} - \int_0^{\pi/4} \tan^2 \theta d\theta \\ &= \frac{\tan^3 \theta}{3} \Big|_0^{\pi/4} - \int_0^{\pi/4} (\sec^2 \theta - 1) d\theta = \left[ \frac{\tan^3 \theta}{3} - \tan \theta + \theta \right] \Big|_0^{\pi/4} = \frac{1}{3} - 1 + \pi/4. \end{aligned}$$

Use  $\sin^2 \theta = 1 - \cos^2 \theta$  and the substitution  $u = \cos \theta$  to get:

$$\begin{aligned} \int_0^{\pi/6} \sin^3 \theta \cos^4 \theta d\theta &= \int_0^{\pi/6} \sin \theta (1 - \cos^2 \theta) \cos^4 \theta d\theta \\ &= - \int_1^{\sqrt{3}/2} [u^4 - u^6] du = \int_{\sqrt{3}/2}^1 [u^4 - u^6] du \\ &= \left[ \frac{u^5}{5} - \frac{u^7}{7} \right] \Big|_{\sqrt{3}/2}^1 = \frac{2}{25} - \sqrt{3} \frac{117}{4480} \end{aligned}$$

6.  $0.171717\dots = \frac{17}{100} \left[ \sum_0^\infty \left( \frac{1}{100} \right)^n \right] = \frac{17}{100} \frac{1}{1-(1/100)} = \frac{17}{99}$  Thus  $2.1212\dots = 2 + \frac{17}{99} = \frac{215}{99}$ .

7. (a) By L'Hopital's rule,  $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{2x} = 0$ .

(b) It is also the case by L'Hopital's rule that  $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = 0$ . Since

$$\lim_{n \rightarrow \infty} \frac{(\ln n)/n^2}{1/n^{3/2}} = \lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = 0,$$

and since  $\sum_1^\infty \frac{1}{n^{3/2}}$  converges, it follows by the limit comparison criterion that  $\sum_2^\infty \frac{\ln n}{n^2}$  converges.

Or one can note that  $(\ln x)/x^2$  is decreasing for  $x \geq 2$  and, by integration by parts

$$\begin{aligned} \int_2^\infty \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \left[ -\frac{\ln x}{x} \Big|_2^b + \int_2^b \frac{1}{x^2} dx \right] \\ &= \frac{\ln 2}{2} + \int_2^\infty \frac{1}{x^2} dx \end{aligned}$$

The last integral converges, so by the integral test, so too does  $\sum_2^\infty \frac{\ln n}{n^2}$ .

8. (a) The best approximation is  $T_2(3.3)$ . Since  $T_2(x) = 1 - 2(x - 3) + (20/2)(x - 3)^2$ , We have  $T_3(3.3) = 1 - 0.6 + 10(.09) = 1.3$ .

(b)  $|f(x) - T_2(x)| \leq \frac{24}{3!} |x - 3|^3 = 4|x - 3|^3$ . Therefore, we shall have  $|f(x) - T_3(x)| < 1/10$  if  $4|x - 3|^3 \leq 1/10$ . This requires  $|x - 3| \leq (40)^{-1/3}$  or  $3 - (40)^{-1/3} \leq x \leq 3 + (40)^{-1/3}$ .

$$\begin{aligned} \mathbf{9. (a)} \quad \cos(3x)e^{x/2} &= \left[1 - \frac{9x^2}{2} + \dots\right] \left[1 + \frac{x}{2} + \frac{(x/2)^2}{2} + \dots\right] \\ &= 1 + \frac{1}{2}x - \frac{35}{8}x^2 + \dots \end{aligned}$$

$$\begin{aligned} \mathbf{(b)} \quad \cos(3x) - e^{x/2} &= \left[1 - \frac{9x^2}{2} + \dots\right] - \left[1 + \frac{x}{2} + \frac{(x/2)^2}{2} + \dots\right] \\ &= -\frac{1}{2}x - \frac{37}{8}x^2 + \dots \end{aligned}$$

$$\mathbf{10.} \quad \frac{1}{3+2x^3} = \frac{1}{3} \frac{1}{1 - (-2x^3/3)} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n (2x^3/3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}} x^{3n}, \text{ and}$$

this converges only if  $|2x^3/3| < 1$ , or  $|x| < (3/2)^{1/3}$ . The next part of the problem does not make sense. It should ask instead for a power series representation of  $\int_0^x f(t) dt$ . We obtain this by integrating term by term:

$$\int_0^x \frac{1}{3+2t^3} dt = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}} \int_0^x t^{3n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}(3n+1)} x^{3n+1}.$$

The radius of convergence of this power series is the same as for the power series of the integrand, namely  $R = (3/2)^{1/2}$ .

For the last part of the problem we set  $x = 1$  in the previous formula to get

$$\int_0^1 \frac{1}{3+2t^3} dt = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}(3n+1)}.$$

Since this is an alternating series with terms decreasing in absolute value, the error made in approximating the integral by the third partial sum of the series is bounded by the absolute value of the fourth term, namely  $\frac{2^4}{3^5 \cdot 13}$ .

**11. (a)** This series diverges because  $\lim_{n \rightarrow \infty} \frac{n}{\ln n} \neq 0$ .

**(b)** This series converges absolutely because  $\left| \frac{(-1)^n n}{n^3 + 4} \right| < \frac{1}{n^2}$  and  $\sum_1^{\infty} (1/n)^2$  converges.

**(c)** This series converges absolutely by the ratio test, because  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 4/5 < 1$ .

**12. (a)** Since  $\frac{1}{(x+3)(x+1)} < \frac{1}{x^2}$  for  $x > 0$ , and since  $\int_1^{\infty} \frac{1}{x^2} dx$  converges, it follows that  $\int_1^{\infty} \frac{1}{(x+3)(x+1)} dx$  converges. Since  $0 < \frac{1}{(x+3)(x+1)} \leq \frac{1}{3}$

for all  $x > 0$ , it follows that  $\int_0^1 \frac{1}{(x+3)(x+1)} dx$  exists and is finite. Therefore  $\int_0^\infty \frac{1}{(x+3)(x+1)} dx = \int_0^1 \frac{1}{(x+3)(x+1)} dx + \int_1^\infty \frac{1}{(x+3)(x+1)} dx$  converges.

(b) Because  $0 < \frac{1}{4+x^2} < \frac{1}{4} = \frac{1}{2^2}$  for all  $x > 0$  and because  $0 < \frac{1}{4+x^2} < \frac{1}{x^2}$ , it follows that  $\int_0^1 \frac{1}{(4+x^2)^{3/2}} dx$  exists and is finite and that  $\int_1^\infty \frac{1}{(4+x^2)^{3/2}} dx$  converges. Hence, as in part (a),  $\int_0^\infty \frac{1}{(4+x^2)^{3/2}} dx$  converges,

**13.** (a) By partial fractions,  $\frac{1}{(x+1)(x+3)} dx = \frac{1}{2} \left[ \frac{1}{x+1} - \frac{1}{x+3} \right]$ . Note: Neither  $\int_0^\infty \frac{1}{x+1} dx$  nor  $\int_0^\infty \frac{1}{x+3} dx$  converges so we cannot proceed by writing  $\int_0^\infty \frac{1}{(x+3)(x+1)} dx = \frac{1}{2} \left[ \int_0^\infty \frac{1}{x+1} dx - \int_0^\infty \frac{1}{x+3} dx \right]$ . Instead, we proceed from the definition:

$$\begin{aligned} \int_0^\infty \frac{1}{(x+3)(x+1)} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{1}{(x+3)(x+1)} dx \\ &= \lim_{b \rightarrow \infty} \frac{1}{2} [\ln(x+1) - \ln(x+3)] \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \frac{1}{2} \ln \left[ \frac{b+1}{b+3} \right] + \frac{1}{2} \ln 3 \\ &= \frac{1}{2} \ln \left[ \lim_{b \rightarrow \infty} \frac{b+1}{b+3} \right] + \frac{\ln 3}{2} = \frac{\ln 1}{2} + \frac{\ln 3}{2} = \frac{\ln 3}{2} \end{aligned}$$

(b) We use trigonometric substitution with  $\tan \theta = x/2$ , so that  $dx = 2 \sec^2 \theta$  and  $\frac{1}{(4+x^2)^{1/2}} = (\cos \theta)/2$ . When  $x = 0$ ,  $\theta = 0$  and as  $x \uparrow \infty$ ,  $\theta \uparrow \pi/2$ . Thus,

$$\int_0^\infty \frac{1}{(4+x^2)^{3/2}} dx = \frac{1}{4} \int_0^{\pi/2} \cos^3 \theta \sec^2 \theta d\theta = \frac{1}{4} \int_0^{\pi/2} \cos \theta d\theta = \frac{1}{4}.$$

**14.** Using  $\cos^2(1/(n+1)) = 1 - \sin^2(1/(n+1))$ ,

$$\sum_{n=1}^N \left[ \sin^2 \frac{1}{n} + \cos^2 \frac{1}{n+1} - 1 \right] = \sum_{n=1}^N \left[ \sin^2 \frac{1}{n} - \sin^2 \frac{1}{n+1} \right]$$

$$\begin{aligned}
&= \sin^2(1) - \sin^2(1/2) + \sin^2(1/2) - \sin^2(1/3) + \cdots \\
&\quad - \sin^2(1/N) + \sin^2(1/N) - \sin^2(1/(N+1)) = \sin^2(1) - \sin^2(1/(N+1))
\end{aligned}$$

Since  $\lim_{x \rightarrow 0} \sin x = \sin 0 = 0$ ,

$$\begin{aligned}
\sum_{n=1}^{\infty} \left[ \sin^2 \frac{1}{n} + \cos^2 \frac{1}{n+1} - 1 \right] &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \left[ \sin^2 \frac{1}{n} - \sin^2 \frac{1}{n+1} \right] \\
&= \lim_{N \rightarrow \infty} \left[ \sin^2(1) - \sin^2(1/(N+1)) \right] = \sin^2(1)
\end{aligned}$$

**15.** (a) The radius of convergence is determined by the set of  $x$  where

$$1 > \lim_{n \rightarrow \infty} \frac{(-1)^{n+1} x^{n+1} / \ln(n+3)}{(-1)x^n / \ln(n+3)} = |x| \lim_{n \rightarrow \infty} \frac{\ln(n+2)}{\ln(n+3)} = |x|$$

Thus the radius of convergence is 1 and the series converges absolute for  $|x| < 1$ . At  $x = 1$ , the series evaluates to  $\sum_0^{\infty} (-1)^n \frac{1}{\ln(n+2)}$  and this converges by the alternating series criterion, because  $\frac{1}{\ln(n+2)}$  decreases to 0 as  $n \rightarrow \infty$ . However at  $x = -1$ , the series evaluates to  $\sum_0^{\infty} \frac{1}{\ln(n+2)}$  and this diverges. To see this, observe that  $\ln x < x$  for  $x > 0$ ; therefore  $\frac{1}{\ln(n+2)} > \frac{1}{n+2}$ . Since  $\sum_0^{\infty} \frac{1}{n+2}$  diverges, it follow from the comparison method that  $\sum_0^{\infty} \frac{1}{\ln(n+2)}$  diverges also. This means also that the series at  $x = 1$  is not absolutely convergent and is hence conditionally convergent.

(b) The radius of convergence in this example is 10. Since the power series is centered at  $c = -1$ , the series converges absolutely for  $-11 < x < 9$ . At  $x = 9$ , the series evaluates to  $\sum_1^{\infty} \frac{1}{n^3}$  which converges absolutely. At  $x = -11$ , the series evaluates to  $\sum_1^{\infty} \frac{(-1)^n}{n^3}$ , which also converges absolutely.

**16.** Recall that the slope of the tangent line to a parametric curve as  $t$  passes through  $t_0$  is  $\frac{y'(t_0)}{x'(t_0)}$ . For this problem,  $x' = -4t$  and  $y' = \pi \cos(\pi t)$ . The parametric equations pass through the origin at  $t = 2$  and  $t = -2$  only. The tangent at  $t = 2$  thus has slope  $\pi \cos(2\pi)/(-8) = -\pi/8$ ; the tangent corresponding to  $t = -2$  has slope  $\pi/8$ . Thus the equations of the tangent lines are  $y = -(\pi/8)x$  and  $y = (\pi/8)x$ .

**17. Solution A:** (Using little calculus) Since  $x(t) = y(t)$  for all  $t$ , the curve  $C$  lies in the straight line  $x = y$ . At  $t = 0$ ,  $(x(0), y(0)) = (0, 0)$  and again at  $t = \pi$ ,  $(x(\pi), y(\pi)) = (0, 0)$ . As  $t$  from 0 to  $\pi$ ,  $x(t) = y(t)$  first increases to a maximum value and then decreases back to the origin. This maximum occurs at  $t$  such that  $0 = x'(t) = 2e^t(\sin t + \cos t)$ , namely at  $t = 3\pi/4$ . At this point  $(x(3\pi/4), y(3\pi/4)) = (\sqrt{2}e^{3\pi/4}, \sqrt{2}e^{3\pi/4})$ . The length of the line segment from  $(0, 0)$  to  $(\sqrt{2}e^{3\pi/4}, \sqrt{2}e^{3\pi/4})$  is  $\sqrt{2}(\sqrt{2}e^{3\pi/4}) = 2e^{3\pi/4}$ .  $C$  is the curve which traverses this segment forward and back, so the total distance traveled along  $C$  is  $4e^{3\pi/4}$ .

We can also do this problem by using the formula  $L = \int_0^\pi \sqrt{(x'(t))^2 + (y'(t))^2} dt$  for the length  $L$  of  $C$ . Since  $x'(t) = y'(t)$ ,  $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{2}|x'(t)| = \sqrt{2}2e^t|\sin t + \cos t|$ . Here is where we need to be careful. Note that  $|\sin t + \cos t| = \sin t + \cos t$  for  $0 \leq t \leq 3\pi/4$ , and  $|\sin t + \cos t| = -\sin t - \cos t$  for  $3\pi/4 \leq t \leq \pi$ . Hence  $|x'(t)| = x'(t)$ , for  $0 \leq t \leq 3\pi/4$  and  $|x'(t)| = -x'(t)$  for  $3\pi/4 \leq t \leq \pi$ . Thus (avoiding integration by parts using the fundamental theorem of calculus)

$$\begin{aligned} L &= \int_0^{3\pi/4} \sqrt{2}x'(t) dt - \sqrt{2} \int_{3\pi/4}^\pi x'(t) dt \\ &= \sqrt{2} [x(3\pi/4) - x(0) - (x(\pi) - x(3\pi/4))] \\ &= 2\sqrt{2} \cdot x(3\pi/4) = 2\sqrt{2} \cdot 2e^{3\pi/4} \sin(3\pi/4) = 4e^{3\pi/4} \end{aligned}$$