FINAL EXAM PRACTICE PROBLEMS

The following problems are generally similar to the type of problems you can expect to find on the final exam. However, not all types of problems are included and it is possible that the final exam will contain some problems quite different from any here.

1. Solve the initial value problem $y' = \frac{1}{x}y - x^3, \quad y(1) = 0.$

2. Find the implicitly defined solution of the initial value problem

$$\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2y + e^y}, \qquad y(2) = 0.$$

3. Find an approximation to the solution of the initial value problem $y' = t^2 + y^2$, $y(0) = 1$ at $t = .1$ by both Euler's method and the improved Euler's method using a step size $h = .1$.

4. Find the general solution of $\quad y'' - y' - 6y = 2e^{-2x} - x.$

5. One solution of $y'' + 6y' + 9y = 0$ is $y = e^{-3x}$. *Use the method of reduction of order* to find a second solution (don't just write one down). Then solve the initial value problem $y'' + 6y' + 9y = 0$, $y(0) = y'(0) = 2$.

6. A mass of 1/2 slug (weighing 16 lb) is hung from the ceiling via a spring. The spring constant is 5 lb/ft and the spring has unstretched length 18 inches; the mass is subject to a linear damping force whose magnitude (in lb) is 3 times the speed of the mass (in ft/sec).

(a) How far below the ceiling is the equilibrium position of the mass? (b) Suppose that the mass is released from a position 1 foot above its equilibrium position with a velocity of 4 ft/sec directed downward. Find the subsequent motion of the mass. (c) Find the velocity of the mass in (b) at the first time it passes through its equilibrium position.

7. For the equation $dy/dt = f(y) \equiv y - 2\sqrt{y}$, $y \geq 0$:

(a) Sketch the graph of $f(y)$ versus y.

(b) Determine the critical (equilibrium points) and classify each one as asymptotically stable, unstable, or semistable.

(c) Find the eventual value of y (i.e., $\lim_{t\to\infty} y(t)$) if initially (i) $y = 2$, (ii) $y = 5$.

8. Find the general solution of the homogeneous equation $\quad x^2y'' + xy' - y = 0$ and then use variation of parameters to find the general solution of the equation $x^2y'' + xy' - y = x$.

9. (a) Find the general solution of the equations $x' = -2x + y; \quad y' = x - 2y$. (b) Give a careful drawing of the phase plane (xy-plane) for this system, showing various typical and special trajectories.

10. Consider the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 0 & -1 \\ 5 & 2 \end{pmatrix}$.

(a) What is the type and stability of the critical point at the origin? (b) Find a solution (in terms of real functions) which satisfies $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

11. Consider the equations $x' = x(x + y - 4)$, $y' = y(1 + 2x - y)$.

(a) Show that $x = 1$, $y = 3$ is a critical point, find the linearized equations near this point, and thus determine whether or not this critical point is stable, asymptotically stable, or unstable. What is the *type* of this critical point?

(b) Find all other critical points.

(c) Sketch the first quadrant $x \geq 0$, $y \geq 0$ of the phase plane, indicating, by arrows or otherwise, regions where x and y are increasing, x is increasing and y decreasing, etc., and where the trajectories are horizontal and vertical.

(d) For each initial condition below, determine (from your sketch) $\lim_{t\to\infty} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$:

(i) $x(0) = 1$, $y(0) = 1$; (ii) $x(0) = 4$, $y(0) = 1$.

12. (a) Solve $y'' + 2xy' + 4y = 0$ as a series $y = \sum_{n=0}^{\infty} a_n x^n$, finding all coefficients up to order 6 and displaying your final answer in the form $y(x) = a_0 y_0(x) + a_1 y_1(x)$.

(b) Find the solution of the equation which satisfies $y(0) = 3$, $y'(0) = 1$.

13. Suppose a power series $\sum_{n=0}^{\infty} a_n x^n$ is a solution to

$$y'' + xy' + 2y = 0$$

(a) Find the first 4 terms of two independent power series solutions

(b) Find the recurrence relation

(c) Find a formula for a_n in terms of a_0, a_1.

ANSWERS (not guaranteed)

1. $y = (x^4 - x)/3$.
2. $x^2y^2/2 - x^3/4 + e^y + 1 = 0$.
3. Euler's method $y(.1) \approx 1.1$, Improved Euler $y(.1) \approx 1.111$.
4. $y(x) = c_1e^{-2x} + c_2e^{3x} - (2x/5)e^{-2x} + x/6 - 1/36$.
5. General solution: $y = c_1e^{-3x} + c_2xe^{-3x}$. Solution of IVP: $y = 2e^{-3x} + 8xe^{-3x}$.
6. 56.4 inches from ceiling. $u(t) = e^{-3t}[\sin(t) - \cos(t)]$, $u'(\pi/4) = \sqrt{2}e^{-3\pi/4}$.
7. Critical points $y = 0$ (stable), $y = 4$ (unstable). For $y(0) = 2$, $\lim_{t\to\infty} y(t) = 0$. For $y(0) = 5$, $\lim_{t\to\infty} y(t) = \infty$,
8. $y = c_1x + c_2/x$ (homogeneous), $y = c_3x + c_2/x + (x/2)\ln x$ (full equation).
9. $x(t) = c_1e^{-3t} + c_2e^{-t}$, $y(t) = -c_1e^{-3t} + c_2e^{-t}$.
10. unstable spiral; $\mathbf{x}(t) = e^t \begin{pmatrix} \cos(2t) - (3/2)\sin(2t) \\ 2\cos(2t) + (7/2)\sin(2t) \end{pmatrix}$

11. $x' = y' = 0$ at $(1, 3)$. Linearized equations: $x' = x + y, \quad y' = 6x - 3y$. Eigenvalues are $-1 \pm \sqrt{10}$; unstable saddle point. Other critical points: $(0, 0)$, $(0, 1)$, $(4, 0)$. Initial condition (i) $(x(t), y(t)) \to (0, 1)$, Initial condition (ii) $(x(t), y(t)) \to (\infty, \infty)$.
12. (a) $y = a_0[1 - 2x^2 - 4x^4/3 + 8x^6/15] + a_1[x - x^3 + x^5/2]$,
    (b) $y = 3[1 - 2x^2 - 4x^4/3 + 8x^6/15] + [x - x^3 + x^5/2]$.
13. (a) $y_1(x) = 1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6 + \cdots$, $y_2(x) = x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - 1/48x^7 + \cdots$
    (b) $a_{n+2} = -a_n/(n+1)$
    (c) $a_{2n} = (-1)^n a_0/(1 \cdot 3 \cdot 5 \cdots 2n - 1), a_{2n+1} = (-1)^n a_1/(1 \cdot 2 \cdot 4 \cdots 2n)$