FINAL EXAM PRACTICE PROBLEMS

The following problems are generally similar to the type of problems you can expect to find on the final exam. However, not all types of problems are included and it is possible that the final exam will contain some problems quite different from any here.

1. Solve the initial value problem $y' = \frac{1}{x}y - x^3, \quad y(1) = 0.$

2. Find the implicitly defined solution of the initial value problem

$$\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2y + e^y}, \qquad y(2) = 0.$$

3. Find an approximation to the solution of the initial value problem $y' = t^2 + y^2, \quad y(0) = 1$ at $t = 0.3$ by Euler's method using a step size $h = 0.1$.

4. Find the general solution of $y'' - y' - 6y = 2e^{-2x} - x$.

5. One solution of $y'' - \tan x \; y' + 2y = 0$ is $y_1 = \sin x$. Find another solution which is linearly independent from this one.

6. A mass of 1/2 slug (weighing 16 lb) is hung from the ceiling via a spring. The spring constant is 5 lb/ft and the spring has unstretched length 18 inches; the mass is subject to a linear damping force whose magnitude (in lb) is 3 times the speed of the mass (in ft/sec).

(a) How far below the ceiling is the equilibrium position of the mass? (b) Suppose that the mass is released from a position 1 foot above its equilibrium position with a velocity of 4 ft/sec directed downward. Find the subsequent motion of the mass. (c) Find the velocity of the mass in (b) at the first time it passes through its equilibrium position.

7. For the equation $dy/dt = f(y) \equiv y - 2\sqrt{y}$, $y \geq 0$:

(a) Sketch the graph of $f(y)$ versus y.

(b) Determine the critical (equilibrium points) and classify each one as asymptotically stable, unstable, or semistable.

(c) Find the eventual value of y (i.e., $\lim_{t\to\infty} y(t)$) if initially (i) $y = 2$, (ii) $y = 5$.

8. Find the general solution of the homogeneous equation $x^2y'' + xy' - y = 0$ and then use variation of parameters to find the general solution of the equation $x^2y'' + xy' - y = x$.

9. (a) Find the general solution of the equations $x' = -2x + y; \quad y' = x - 2y$. (b) Give a careful drawing of the phase plane (xy-plane) for this system, showing various typical and special trajectories.

10. Consider the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 0 & -1 \\ 5 & 2 \end{pmatrix}$.

(a) What is the type and stability of the critical point at the origin? (b) Find a solution (in terms of real functions) which satisfies $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (there should be no i in your final answer).

11. (a) Solve $y'' + 2xy' + 4y = 0$ as a series $y = \sum_{n=0}^{\infty} a_n x^n$, finding all coefficients up to order 6 and displaying your final answer in the form $y(x) = a_0 y_0(x) + a_1 y_1(x)$.

(b) Find the solution of the equation which satisfies $y(0) = 3$, $y'(0) = 1$.

12. (a) Show that $x = 0$ is a regular singular point for $xy'' + y = 0$, and find the exponents r_1 and r_2 at the singular point $x = 0$.

(b) Find the first three nonzero terms in each of two linearly independent solutions about $x = 0$.

13. Find the solution of the given initial value problem: $y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t)$, $y(0) = 0$, $y'(0) = 0$.

ANSWERS (not guaranteed)

1. $y = (x^4 - x)/3$.
2. $x^2y^2/2 - x^3/4 + e^y + 1 = 0$.
3. Euler's method $y(.3) \approx 1.1$.
4. $y(x) = c_1e^{-2x} + c_2e^{3x} - (2x/5)e^{-2x} + x/6 - 1/36$.
5. $y_2(x) = -1 + \sin x \ln(\sec x + \tan x)$.
6. 56.4 inches from ceiling. $u(t) = e^{-3t}[\sin(t) - \cos(t)]$, $u'(\pi/4) = \sqrt{2}e^{-3\pi/4}$.
7. Critical points $y = 0$ (stable), $y = 4$ (unstable). For $y(0) = 2$, $\lim_{t\to\infty} y(t) = 0$. For $y(0) = 5$, $\lim_{t\to\infty} y(t) = \infty$,
8. $y = c_1x + c_2/x$ (homogeneous), $y = c_3x + c_2/x + (x/2)\ln x$ (full equation).
9. $x(t) = c_1e^{-3t} + c_2e^{-t}$, $y(t) = -c_1e^{-3t} + c_2e^{-t}$.
10. unstable spiral; $\mathbf{x}(t) = e^t \begin{pmatrix} \cos(2t) - (3/2)\sin(2t) \\ 2\cos(2t) + (7/2)\sin(2t) \end{pmatrix}$
11. (a) $y = a_0[1 - 2x^2 - 4x^4/3 + 8x^6/15] + a_1[x - x^3 + x^5/2]$,
    (b) $y = 3[1 - 2x^2 - 4x^4/3 + 8x^6/15] + [x - x^3 + x^5/2]$.
12. (a) $r_1 = 1$, $r_2 = 0$,
    (b) $y_1(x) = x - x^2/2 + x^3/12 + ...$,
    $y_2(x) = -y_1(x)\ln x + 1 - 3x^2/4 + 7x^3/36 + ...$
13. $y = u_{\pi/2}(t)[1 - \cos(t - \pi/2)] + 3u_{3\pi/2}(t)\sin(t - 3\pi/2) - u_{2\pi}(t)[1 - \cos(t - 2\pi)]$.