Mathematics 244 Essay 4

Series Methods

Fall 2003

Introduction. The fourth segment of the course works through Chapter 5 of the text on Series Solutions of Differential Equations. All equations dealt with in the text are **linear** and **homogeneous** and most have coefficients that are polynomials in the independent variable (which is usually x here). The examples are all **second order** to remove the temptation to get a closed-form solution by the methods of Chapter 1 without introducing excessive complications. Many of these equations have been found to have useful applications and are named after someone who determined interesting properties of the equation or its solutions.

Although **only** linear homogeneous equations are studied in this course, series solutions of nonlinear equations can be found in a similar way, and the solution in the linear case may become clearer if it is extended to an **undetermined coefficients** method for inhomogeneous equations.

Ordinary points. If the equation satisfies the hypothesis of the Existence and Uniqueness Theorem at a point, that point is called **ordinary**. Other points are called **singular**. For second order linear equations whose coefficients are polynomials, the only singular points are the roots of the coefficient of d^2y/dx^2 (called the **leading coefficient** of the equation).

In almost all examples, our solutions will be assumed to be of the form

$$y = \sum_{k=0}^{\infty} a_k x^k. \tag{1}$$

Such **power series** are known to converge in sets of the form

$$\{\, x : |x| < r \,\}$$

that are symmetric intervals around $x = 0$. It is useful to allow **complex** values of x, and this property persists to obtain a **circle** of convergence in the complex plane. We make no direct use of this, but we present the following fact: the series solution (1) of a differential equation **converges in the largest circle around the origin that does not include a singular point of the equation**. In particular, if the leading coefficient is a constant, all points are ordinary and the series has an infinite radius of convergence, which requires that $|a_k|$ approaches zero **rapidly** — the series for e^x is a typical example.

If it is ever necessary to consider series in powers of $(x - x_0)$ to describe solutions with initial conditions at points $x = x_0$ with $x_0 \neq 0$, it will usually be best to make an **explicit** change of variable $u = x - x_0$ **in the equation**. Since $dx/du = 1$, we have $dy/du = dy/dx$, so this process is only an algebraic change of variables on the coefficients.

If the leading coefficient is $1 + x^2$, there are no real singular points, but $x = \pm i$ are singular. The series (1) will have radius of convergence 1. If the series obtained for the solution with particular initial conditions at $x = 0$ is used to find the value of the solution and its derivative at $x = 1/2$ (which is within the interval of convergence), those values could be used to find a series represent **the same** solution as a series in powers of $(x - 1/2)$. The distance from $1/2$ to $\pm i$ is $\sqrt{5}/2$, so this series will converge as far as $(1 + \sqrt{5})/2$.

Initial conditions. For second order equations at an ordinary point, there will be a unique solution to the equation for any **initial conditions** giving values of $y(0)$ and $y'(0)$. Substituting $x = 0$ into the series (1) tells us that $a_0 = y(0)$. Differentiating (1) term-by-term and substituting $x = 0$ into the result tells us that $a_1 = y'(0)$. (While the validity of these operations requires **proof**, the results are correct for the functions that we meet in this course, allowing series to be manipulated exactly like polynomials with infinitely many terms.) To solve the equation, one simply substitutes the series (1) into the equation and collects all terms of the same degree in the result. The coefficients of this series **must all be zero** for a solution of a homogeneous equations. This gives a sequence of linear equations in the a_k. We will see that the constant term of the result involves only a_0, a_1, and a_2; the linear term involves these together with a_3; and, in general, the coefficient of x^m involves a_k with $0 \le k \le m+2$. In particular, each **one** new term introduces **one** new a_k. This allows all a_k to be found in terms of a_0 and a_1. The connection between the a_k and the standard form of initial conditions tells us that this is the right way to organize the process of solving infinitely many equations. Wherever we stop, we will have all the terms of the series through some degree, and the conventional use of series treats these polynomials as a sequence of approximations to the function represented by the series.

In many cases, the sequence of equations can be described by a single equation involving a parameter that can be solved to get a formula for the a_j. We do not follow this direction in most examples, preferring to concentrate on the obtaining the **numerical** values of the initial terms in the series.

Exploiting linearity. Instead of forming one complicated expression involving the a_k, it is possible to organize the work around substituting only the $y = x^k$ into the equation. In order to describe this approach, we write $L[y]$ for the combination of y and its derivatives set equal to zero in the equation. For example, consider the equation in Exercise 9 of Section 5.2:

$$(1+x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0.$$

Here,

$$L[y] = (1+x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y$$

Then, here are the first few values of $L[x^k]$:

$$\begin{aligned}
L[1] &= 6 \\
L[x] &= \quad 2x \\
L[x^2] &= 2 \\
L[x^3] &= \quad 6x \\
L[x^4] &= \qquad 12x^2 + 2x^4 \\
L[x^5] &= \qquad\qquad 20x^3 + 6x^5
\end{aligned}$$

We have arranged these results with terms of the same degree arranged in columns. Linearity gives the equation

$$L\left[\sum a_k x_k\right] = \sum a_k L[x_k]$$

which tells us to multiply the **rows** by $a_0, a_1, a_2, a_3, a_4, a_5$ and add the **columns** to get the equations in the a_k that say that the series (1) satisfies the differential equation. In this case, we get

$$\begin{aligned}
2a_2 + 6a_0 &= 0 \\
6a_3 + 2a_1 &= 0 \\
12a_4 &= 0 \\
20a_5 &= 0
\end{aligned}$$

In each of these rows, the first term is a multiple of the a_k, starting with a_2, that will be found by solving that equation, and the other terms have smaller subscripts. For this equation, we find that $a_2 = -3a_0$, $a_3 = -a_1/3$, but $a_5 = a_4 = 0$. Additional equations will show that all subsequent $a_k = 0$. All solutions of this equation are polynomials, since the general solution is

$$y = a_0(1 - 3x^2) + a_1(x - x^3/3).$$

Although we only expected a series with a radius of convergence of 1, we found polynomial solutions. The points $x = \pm i$ are still singular, since Abel's theorem tells us that the **Wronskian** of a basis for the solutions will be a constant multiple of $(1 + x^2)^2$, which you could verify directly for this basis.

Parameterizing the solutions in terms of a_0 and a_1 allows the solution satisfying given initial conditions and $x = 0$ to be written immediately.

Solutions will rarely be this simple, but we should organize work so that such good fortune doesn't become an obstacle to getting the correct answer.

If we needed additional rows in this table, they could be derived from the general formula

$$L[x^k] = (k^2 - 5k + 6)x^k + (k^2 - k)x^{k-2}$$

Thus, there are at most two terms, but the second term is zero if $k = 0$ or $k = 1$ and the first term is zero if $k = 2$ or $k = 3$ (these special cases were noted in the discussion above). The process of **adding the columns** involves identifying x^m as appearing only when $m = k$ or $m = k - 2$. The total coefficient is

$$\big((m+2)^2 - (m+2)\big)a_{m+2} + \big(m^2 - 5m + 6\big)a_m.$$

Setting these expressions equal to zero gives the **recurrence formula**

$$a_{m+2} = \frac{(m-2)(m-3)}{(m+2)(m+1)}a_m$$

that computes the a_k in order. In this case, the recurrence tells us that $a_4 = a_5 = 0$ and then all successive $a_m = 0$. In this form, it fails to apply when $m = -1$ or $m = -2$, corresponding to the fact that a_0 and a_1 are allowed to be nonzero while a_m for $m < 0$ must be zero.

A useful family of equations.

This example is a special case of the equation

$$(a + bx^2)\frac{d^2y}{dx^2} + cx\frac{dy}{dx} + (r + sx^2)y = 0. \tag{2}$$

with constants a, b, c, r, s (the gap in naming the coefficients results from avoiding letters that have fixed meaning). The point $x = 0$ is an ordinary point if $a \neq 0$.

Again, the left side will be denoted $L[y]$ and we find

$$L[x^k] = ak(k-1)x^{k-2} + \big(bk(k-1) + ck + r\big)x^k + sx^{k+2}$$

You should not treat this as a **formula** — it merely echos the terms that appear when you compute $L[x^k]$. It is given here to remind you that a term x^m appears in such an expression only if $m = k - 2$, $m = k$ or $m = k + 2$. When $a \neq 0$, the x^{k-2} has a zero coefficient precisely when $k = 0$ or $k = 1$, for which $k - 2$ is negative. There is an additional simplification when $s = 0$ removing the x^{k+2} term. In this case, the recurrence formula gives a_m as an explicit multiple of a_{m-2} for all $m \geq 2$. The values of a_0 and a_1 are found from the initial conditions.

The Hermite equation. Exercise 21 of Section 5.2 deals with the equation

$$y'' - 2xy' + \lambda y = 0$$

which is known as the **Hermite equation**. Here, λ is a constant. The solutions of the equation for various λ will be described. Denoting the left side of this equation by $L[y]$, we find

$$L[x^k] = k(k-1)x^{k-2} + (\lambda - 2k)x^k.$$

In particular,

$$\begin{array}{rcllllll}
L[1] &=& \lambda & & & & & \\
L[x] &=& & (\lambda-2)x & & & & \\
L[x^2] &=& 2 & + & (\lambda-4)x^2 & & & \\
L[x^3] &=& & 6x & + & (\lambda-6)x^3 & & \\
L[x^4] &=& & & 12x^2 & + & (\lambda-8)x^4 & \\
L[x^5] &=& & & & 20x^3 & + & (\lambda-10)x^5
\end{array}$$

Multiplying row k by a_k and adding gives the following results as the sums of the columns

$$2a_2 + \lambda a_0 = 0$$

$$6a_3 + (\lambda-2)a_1 = 0$$

$$12a_4 + (\lambda-4)a_2 = 0$$

$$20a_5 + (\lambda-6)a_3 = 0$$

which are instances of the recurrence formula $m(m-1)a_m + (\lambda - 2m + 4)a_{m-2} = 0$, or

$$a_m = \frac{2m-4-\lambda}{m(m-1)}a_{m-2}.$$

Again, we see a denominator of $m(m-1)$ corresponding to the fact that a_0 and a_1 are determined from initial conditions while $a_m = 0$ for $m < 0$. The numerator shows that one solution will be a polynomial when λ is a positive even integer. For example, when $\lambda = 6$, one has $a_5 = 0$ which leads to $a_m = 0$ for all larger odd numbers. The solution with $a_0 = 0$ and $a_1 = 1$ is $x - (2/3)x^3$, while the solution with $a_0 = 1$ and $a_1 = 0$ is the infinite series

$$1 - 3x^2 + \frac{1}{2}x^4 + \frac{1}{30}x^6 + \frac{1}{280}x^8 + \frac{1}{2520}x^{10} + \dots$$

with $a_0 = 1$

Euler equations. If $a = s = 0$ in (2), $L[x^k] = \big(bk(k-1) + ck + r\big)x^k$, so $y = x^k$ is a solution if $bk(k-1) + ck + r = 0$. This is a quadratic equation in k which usually has two roots. When there are two distinct roots, we have two solutions whose Wronskian is different from zero everywhere except at the singular point $x = 0$.

Since $x = 0$ is a singular point, solutions should be found separately for $x > 0$ and $x < 0$. As noted in the textbook, the solutions $y = x^k$ are easily interpreted when $x > 0$, and the corresponding functions of $-x$ should be used when $x < 0$. The interpretation is based on writing $x^k = e^{k \ln x}$. The similarity with the method of solution in the constant coefficient case is explained by the change of variables $x = e^t$. This is outlined in Exercise 23 of Section 5.5.

Regular Singular points. If an equation can be written as the sum of an Euler equation and some higher order terms, it is said to have a **regular singular point** at $x = 0$. There are series solutions in this case, but the lowest degree term of the series must be one of the solutions of the Euler equation part of the equation. Our special form (2) is a second order equation with a regular singular point is $a = 0$ and $b \neq 0$.

To illustrate the method, consider Problem 1 of Section 5.6:

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$$

This does not appear to be in the form (2), but has that form if multiplied by x, so the results will have the properties associated with that form, although the details will look a little different while working with the form in which the equation is presented.

As usual, the left side of the equation will be denoted $L[y]$. The main technique if to find $L[x^k]$:

$$L[x^k] = 2k(k-1)x^{k-1} + kx^{k-1} + x^{k+1} = (2k^2 - k)x^{k-1} + x^{k+1} = k(2k-1)x^{k-1} + x^{k+1}.$$

Constant multiples of x^m appear if $m = k+1$ or $m = k-1$. That is, $k = m-1$ or $k = m+1$. If $k = m-1$, the term is x^m; if $k = m+1$, the term is $(m+1)(2m+1)x^m$. If we have a solution that is a sum of terms $a_k x^k$, then

$$a_{m-1} + (m+1)(2m+1)a_{m+1} = 0.$$

Note that this notation is **different from that used in the text**. We use indices that **agree with the exponent** on x without first separating out the lowest degree term. From this recurrence we see that it is possible for $a_{m+1} \neq 0$ while $a_{m-1} = 0$ if $(m+1)(2m+1) = 0$. This selects the possible lowest degree terms of $m+1 = 0$ or $m+1 = 1/2$. The recurrence will give the value of $a_2, a_4, a_6 \ldots$ as multiples of a_0 and $a_{5/2}, a_{9/2}, a_{13/2}, \ldots$ as multiples of $a_{1/2}$. Taking one of these leading coefficients to be 1 and the other 0 gives solutions

$$y_1 = 1 - \frac{1}{6}x^2 = \frac{1}{168}x^4 - \frac{1}{11088}x^6 + \ldots$$

and

$$y_2 = x^{1/2} - \frac{1}{10}x^{5/2} + \frac{1}{360}x^{9/2} - \frac{1}{28080}x^{13/2} + \ldots$$

Inhomogeneous equations. The text limits itself to homogeneous equations in this chapter, but similar methods apply to inhomogeneous equations. The broader view may help to clarify the role of the exponents that appear in the solutions.

A **differential operator** $L[y]$ typically expresses $L[x^k]$ as a polynomial (or series) in which the lowest power of x that appears is x^{k-n} for some n. In particular, $n = 2$ when $x = 0$ is an ordinary point, $n = 0$ for the usual equation with a regular singular point, and $n = 1$ for the equation considered in the previous section. However, there are special values of k for which the coefficient of this term is zero. For ordinary points, these values are **always** $k = 0$ and $k = 1$. For regular singular points, they are the powers that satisfy the associated Euler equation. For these values of k, we do not have an expression ϕ for which $L[\phi]$ has k^{k-n} as its lowest degree term, although we will find such an expression soon.

If we want to solve the inhomogeneous equation $L[y] = \alpha(x)$ where $\alpha(x)$ is given as a sum of constant multiples of powers of x, we can find ϕ such that $L[\phi]$ has lowest degree term equal to the term of lowest degree in $\alpha(x)$. This allows our equation to be written

$$L[y - \phi] = \alpha(x) - L[\phi]$$

where the right side now has its **lowest degree term** of **higher degree** than the lowest degree term in $\alpha(x)$. If we **never meet** one of our special exponents, this process determines a series solution of the inhomogeneous equation. The steps in this process always increase the index by an integer.

The special values of k have the property that $L[x^k]$ has only terms of higher degree than expected. Except in special cases, this means that $\alpha(x) = -L[x^k]$ is an expression for which the inhomogeneous equation has a series solution ϕ. In this case, $x^k + \phi$ is a solution of the **homogeneous** equation $L[y] = 0$. This is the same solution that was obtained using the recurrence formula. These special $L[x^k]$ cause a disturbance in the sequence of exponents of lowest degree terms: some values are skipped, and others are duplicated. The duplicated values allow two different solutions to some inhomogeneous equations, whose difference is a solution of the homogeneous equation. This happens at most twice for a second order equation.

This can fail to find a solution to the homogeneous equation if the duplicated degree of one $L[x^k]$ meets a skipped degree while constructing the series. This only happens if the skipped degree exceeds the duplicated degree by an integer. A repeated root of the equation determining the special degrees also fails to find a second solution of the homogeneous equation.

If $y_1(x)$ is a solution of the homogeneous equation $L[y] = 0$, then skipped degrees can be filled in by considering $\phi = y_1(x)\ln x$, since

$$\frac{d}{dx}\big(y_1(x)\ln x\big) = y_1'(x)\ln(x) + \frac{y_1(x)}{x}$$

and

$$\left(\frac{d}{dx}\right)^2\big(y_1(x)\ln x\big) = y_1''(x)\ln(x) + 2\frac{y_1'(x)}{x} - \frac{y_1(x)}{x^2}.$$

From this, it follows that the terms containing $\ln x$ disappear from $L[y_1(x)\ln x]$ and the remaining terms give a term of the **expected** lowest degree in $L[y_1]$. As an example, consider Exercise 3 of Section 5.6, which has $L[y] = xy'' + y$. Then,

$$L[x^k] = k(k-1)x^{k-1} + x^k,$$

whose lowest degree term usually has degree $k-1$. The special values of k are $k = 0$ and $k = 1$ — just like the case of an ordinary point, although $x = 0$ is singular in this case. Here, $L[1] = 1$ and $L[x] = x$. The next case, $L[x^2] = 2x + x^2$, is typical of all other cases. From this, we see that -1 is a missing minimal degree, and 1 is duplicated. The recurrence formula is $a_m + m(m+1)a_{m+1} = 0$, so starting with $a_1 = 1$ leads to

$$\begin{aligned} y_1(x) &= x - \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \dots \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k!(k-1)!} \end{aligned}$$

Then,

$$\begin{aligned} y_1'(x) &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{kx^{k-1}}{k!(k-1)!} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(n!)^2} \end{aligned}$$

with $n = k - 1$, and $y_1''(x) = -y_1(x)/x$. Then

$$\begin{aligned} L[y_1(x)\ln x] &= xy_1''(x)\ln(x) + y_1(x)\ln x + 2y_1'(x) - \frac{y_1(x)}{x} \\ &= 2y_1'(x) - \frac{y_1(x)}{x} \\ &= 1 + \frac{1}{4}x + \dots \end{aligned}$$

This duplicates degree 0, so there is a power series ψ, which may taken without a term of degree 1 such that $L[\psi] = L[y_1(x)\ln x]$. If we take $y_2(x) = \psi - y_1(x)\ln x$, then $L[y_2] = 0$ and $L[y_2(x)\ln x]$ involves only powers of x with lowest exponent -1. Inhomogeneous equations can be solved as a series supplemented by $y_1(x)\ln x$ and $y_2(x)\ln x$. When these new quantities are added, they should **replace** $x^0 = 1$ and $x^1 = x$ whose role was to be the initial terms of the series y_1 and y_2 that gave solutions of the homogeneous equation.

End of Chapter 5 overview