Mathematics 244 Summary 1

Solution in Closed Form

Fall 2003

0. Introduction.

Calculus has always emphasized the **elementary functions**: polynomials, exponentials, logarithms and the trigonometric functions. Most of the work in this course follows the same pattern, although the **theory** is based on the same general idea of function that provides the foundation of the calculus, and some new functions are introduced by describing differential equations that they satisfy. Equations with familiar solutions have one big advantage — **it is easy to check answers** — and one big disadvantage — **bad habits in algebra and calculus lead to wrong answers**. The advantage can minimize the damage done by the disadvantage — **if you form the habit of checking your answers**. The main theorem of the subject asserts that an **initial value problem** (i.e., a differential equation together with the value of the solution at one point) has a unique solution unless the problem has bad behavior of a rather limited type. Thus, if you verify that you have **a solution**, then you have **the solution**. This means that differential equations can be solved by guessing the solution and checking that your guess is correct. A slight generalization of this method is to guess the **form** of the solution, and use the check to evaluate the parameters that express individual functions of this form.

This **guess and check** method (which should not be called “trial and error”, because you aim to avoid making any errors) is most useful in solving linear equations. It will play a smaller role in this Summary, where nonlinear equations are emphasized. However, one should avoid considering the methods introduced to solve equations as formulas. Any attempt to simplify the expressions used to summarize the method relies on techniques of algebra and calculus that are most likely to be done wrong. On the other hand, the process of checking an answer usually only requires the ability to differentiate elementary functions and check that two algebraic expressions are equal. Such operations are more **robust** (i.e., likely to be done correctly) in hand computation.

1. Integrals.

The simplest form of differential equation has the form

$$\frac{dy}{dt} = f(t).$$

The solution to such an equation is the **indefinite integral** of $f(t)$, and the “$+C$” that you were taught to write after evaluating the integral allows you to satisfy initial conditions.

For example: when restricted to positive values of t, the equation $dy/dt = 1/t$ has the solution $y = (\ln t) + C = \ln(kt)$ where $C = \ln k$ or $k = e^C$. Restricting to positive values of t is one way to deal with the fact that $1/t$ is not defined at $t = 0$. When solving a differential equation, the solution is only valid at points that can be reached from the initial point without passing through any point where the equation has bad behavior. In this case, we are unable to define the expression that is supposed to represent dy/dt, so we expect trouble. Indeed, all solutions approach $-\infty$ as $t \to 0$. Evaluating this solution at $t = 1$ shows that $y(1) = C$, so the form involving C is most natural for dealing with initial value problems in which the value of $y(1)$ is given. In other cases, it may be more useful to write the solution in the second form with a positive value of k. Note that $y = \ln(kt)$ appears to be a solution for **all** k. If $k > 0$, the function is defined only for $t > 0$; if $k < 0$, the function is defined only for $t < 0$; and if $k = 0$, the function isn't defined at all. The second form of the solution allows us to get around the special assumption made to find a solution by integration and discover the solutions valid for negative t.

Some functions defy all the **techniques of integration** introduced in the second semester of calculus. These can be used to define new functions. If a new function is to be defined, it must include a recipe for evaluating the function **everywhere** — it is not enough to have the function defined up to "$+C$". The notation to express this as an integral can be awkward. For example, as used in Maple,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

This definition can also be expressed slightly more conveniently as the solution of the initial value problem

$$\frac{dy}{dx} = \frac{2}{\sqrt{\pi}} e^{-x^2} \quad \text{with} \quad y(0) = 0.$$

The scale factor $2/\sqrt{\pi}$ is introduced to give

$$\lim_{x \to +\infty} \operatorname{erf}(x) = 1.$$

Although this method of simplifying the definition of functions defined as integrals has some value, it is only a small part of the ability of differential equations to characterize functions.

2. Inverse functions.

The **inverse function theorem** says that

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1,$$

wherever both are defined. This was used in elementary calculus. For example, if $y = \ln x$, then $dy/dx = 1/x$. Hence, $dx/dy = x$. The inverse function of the logarithm is the exponential, so $x = e^y$ and $dx/dy = x = e^y$, which obtains the formulas for the derivative of the exponential from the formula for the derivative of the logarithm. A more elaborate example obtains the derivatives of the inverse trigonometric functions. In the context of differential equations, this allows equations of the form $dy/dx = f(y)$ to be solved. The original form of the solution expresses x in terms of y with a parameter that plays the role of the constant of integration. It is often possible to invert this function to get an explicit solution of the differential equation. This process will cause the parameter to appear in an unusual form in the solution. Even a simple case, like $dy/dx = y$ has solutions $y = Ae^x$ for some constant A. In this case, $A = y(0)$, so it is extremely convenient when solving problems with initial values at $x = 0$.

More elaborate examples are the **population models** of section 2.5 of the textbook. The **logistic equation**

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \tag{2.5.7}$$

is shown to have the **general solution**

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}. \tag{2.5.11}$$

Here, r and K are constants that describe properties of the population being studied, and y_0 is a parameter representing the values of y when $t = 0$. (The equation numbers are those appearing in the text,including the section number, as an aid in locating this example.)

The dependence on the parameter y_0 is not as simple as it was in previous examples.

We also considered the **Gompertz equation**

$$\frac{dy}{dt} = ry\ln(K/y)$$

discussed in **Exercises 16 and 17 of Section 2.5**. **Students should work these exercises in detail**.

The logistic equation has a unique solution through each point (t, y). The Gompertz equation requires $K/y > 0$ (strictly!) for the logarithm to be defined, but otherwise behaves nicely.

In these models, r is a positive constant representing the **growth rate** and K is a positive constant representing the **limiting population**. By examining the equations, one sees immediately that $dy/dt > 0$ for $0 < y < K$, so the solutions will be increasing in this domain. Note that, this is would be harder to discover from the explicit solution of the differential equation. In the usual applications, only values of y between 0 and K are considered, but the equations make good mathematical sense for $y > K$ and lead to solutions that are decreasing. Formula (2.5.11) describes the solution in all cases, although the function given by this formula has a vertical asymptote for some value of t whenever y_0 and $K - y_0$ have opposite signs.

In the course of solving the logistic equation, the textbook obtains formula (2.5.8) expressing the second derivative of y with respect to t. **Students should apply this formula to both the logistic and Gompertz equation**. It should be noted that this formula is nothing more than the **chain rule**.

3. Equilibrium values of autonomous equations.

Equations such as those studied in section 2.2 in which the independent variable doesn't appear are called **autonomous**. If $y = f(t)$ is a solution of $dy/dt = g(y)$, then $y = f(t - t_0)$ is also a solution for and t_0. This may not be the general solution, since different initial values of y may lead to solutions that cannot be related in this way. For example, the solutions with $y > K$ and the solutions with $0 < y < K$ in our population models take no common values.

It is a common feature of all autonomous equations that most solutions either increase for all t or decrease for all t. An equation $dy/dt = g(y)$ says that whether y is an increasing or decreasing function of t depends only on the sign of $g(y)$, which depends only on y. If $g(y)$ is continuous, a place where $g(y) < 0$ and a place where $g(y) > 0$ are separated by a place where $g(y) = 0$. These are lines of **constant** y. Such lines are themselves solutions of the differential equation. The uniqueness theorem implies that solutions of a differential equation **do not cross**, so these lines divide the (t, y) plane into bands and all solutions in one band are **monotonic** from one end of the band to the other. The solutions in each band are t-translates of one another.

The constant solutions of the differential equation are called **equilibrium values**. Behavior of solutions between equilibrium values are determined by the sign of $g(y)$, which can only change at equilibrium values. (Note that it doesn't have to change. $y = 0$ is an equilibrium value of $dy/dt = y^2$ although all solutions other than $y = 0$ are everywhere increasing. **Students should find the general solution of this equation.**)

In Chapter 9, **autonomous systems** appear. There are still equilibrium values, but they no longer form **barriers**: solutions may **move around them** in different ways. An important tool in studying systems is the **linearization** of the system at an equilibrium point. This is also available in the case of the single equation $dy/dt = g(y)$. At any point at which $g(y) = 0$, the value of $g'(y)$ determines the behavior of solutions near that equilibrium point. If $g'(y) > 0$, then $g(y)$ is increasing, so it is negative for smaller y and positive for larger y. The solutions $y(t)$ are decreasing below this equilibrium value and increasing above it. In all cases, they **move away from the equilibrium point**. Such points are called **unstable**. If $g'(y) < 0$ at an equilibrium point, this analysis is reversed, and one finds that all solutions **approach** the equilibrium point, leading to the points being called **stable**. Equilibrium points at which $g'(y) = 0$ do not admit a linearization, and may have different types of behavior. In particular, consider the solutions of $dy/dt = y^2$.

4. Separable equations.

Consider the differential equation

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y} \tag{2.2\#7}$$

Before starting to solve the equation, we should determines where the equation is well behaved. The only obstruction to finding the expression on the right side of the equation is the need to divide by $y + e^y$. This quantity is zero at one particular value of y, (approximately $y = -.5671432904$). Solutions can be found as functions of x except on this line. If y is larger than this, $y + e^y > 0$; if y is smaller, $y + e^y < 0$. Each solution has a **critical point** where $x - e^{-x} = 0$, i.e., at $x \approx 0.5671432904$. If $y + e^y > 0$, this point is a local minimum; if $y + e^y < 0$, this point is a local maximum. Not all solutions are defined at this value of x; some reach the line where $y + e^y = 0$ and cannot continue to be defined as functions of x.

It is also possible to reverse the roles of x and y and write the equation as

$$\frac{dx}{dy} = \frac{y + e^y}{x - e^{-x}}.$$

Now, we see that solutions can be defined as functions of y except where $x - e^{-x} = 0$, and the value where $y + e^y = 0$ identifies the critical point of those solutions that are functions of y defined at this value.

If you plot a **slope field** of this equation, the artificial distinctions forced by requiring the solution to a function of either x or y disappear. Horizontal and vertical tangents are no different from those in other directions. When we consider the **phase plane** of an autonomous system in chapter 9, the curves where the tangents are either horizontal or vertical are called **nullclines**. They correspond to points at which the solutions cannot be given as functions of x or y. This equation can be made into a phase plane by introducing a new variable t such that the derivatives of x and y with respect to t are **everywhere well behaved**. For example, one could take

$$\frac{dx}{dt} = y + e^y$$
$$\frac{dy}{dt} = x - e^{-x}$$

This system has a unique solution for any initial conditions. The lines described above are the nullclines of the system and they intersect at the unique equilibrium point of the system. Linearizing the system at that point reveals that it is a **saddle point**, as is evident from the slope field.

There is another way to rewrite the equation. If we multiply by $y + e^y$, equation 2.2#7 becomes

$$\left(y + e^y\right)\frac{dy}{dx} = x - e^{-x}$$

or

$$\frac{d}{dx}\left(\frac{y^2}{2} + e^y\right) = \frac{x^2}{2} + e^{-x}$$

Since two functions with the same derivative differ by a constant, this allows the general solution to be written as

$$\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C$$

In general, this equation fails to define y as an elementary function of x, but it is **always** a simple matter to test whether two points lie on the curve defined by a single value of C. Such **implicit functions** are usually considered to be solutions of the equation because they have handled all of the calculus. Even when it is possible to solve the equation algebraically, it may be better to leave the solution in the form of an implicit function.

To **do the calculus**, it is usually better to **separate the differentials** so that **all references** to x appear on one side of the equation and **all references** to y on the other. Then, you insert integral signs to turn equality of differentials into equality of integrals. If you insert $+C$ at this step, you may treat this as an answer in which the indefinite integrals are to be replaced by **any** function whose derivative is the integrand. In this example, this gives the answer in the form

$$\int y + e^y\, dy = \int x - e^{-x},\ dx + C.$$

The integration may be done in any way that is convenient. There is no fixed way to organize the rest of the work. Each integral needs to be evaluated separately, using a scratch area if necessary.

The independent variable t can be eliminated from an autonomous system by dividing dy/dt by dx/dt to obtain dy/dx. This leads to a single equation in x and y. An important example is the **predator-prey system**

$$\begin{aligned} \frac{dx}{dt} &= x(a - \alpha y) \\ \frac{dy}{dt} &= y(-c + \gamma x) \end{aligned} \tag{9.5.1}$$

which can be written

$$\frac{dy}{dx} = \frac{y(-c + \gamma x)}{x(a - \alpha y)}$$

Separating the variables then gives the solution

$$\int \frac{a - \alpha y}{y}\, dy = \int \frac{-c + \gamma x}{x}\, dx + C.$$

The integrals are easily evaluated and the solutions may be describe in many forms.

Since the constant parameterizing the general solution may be written in many different forms, two versions of the general solution of the same equation may look different. Checking the solution in the equation will confirm that the equation is satisfied and the details of the check should reveal the relation between different versions of the solution. **Students should find general solutions** of the equations in exercises 2, 5, 11, 17 of section 2.2 of the textbook. Then, **find the solutions with the given initial conditions** for exercises 11 and 17 **in explicit form**.

This approach can also be used to find the equation whose solution is a given family of curves that fill (a portion of) the plane without intersecting (or intersecting in only special ways).

One example is the set of **right-facing tangent lines** to the parabola $y = x^2$. At the point (c, c^2) of the parabola, the tangent line has the equation $y = 2cx - c^2$, since the point satisfies this **linear equation** and slope agrees the value of dy/dx at the point. On each of these lines, $dy/dx = 2c$. Eliminating c between these two equations gives

$$y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2.$$

Rearranging this gives

$$\left(\frac{dy}{dx}\right)^2 - 4x\frac{dy}{dx} + 4y = 0.$$

Solving by the quadratic formula

$$\frac{dy}{dx} = 2x \pm \sqrt{4x^2 - 4y} = 2\big(x \pm \sqrt{x^2 - y}\big).$$

To resolve the sign, return to the equation to obtain

$$x^2 - y = x^2 - 2cx + c^2 = (x - c)^2.$$

Since we have restricted to the portion of the line with $x - c > 0$, it is the **negative** square root that is consistent with $dy/dx = 2c$. The final form of the equation is

$$\frac{dy}{dx} = 2\big(x - \sqrt{x^2 - y}\big).$$

The expression on the right side of this equation is defined only for $y \le x^2$ — below the parabola — but the condition required for a unique solution fails on the parabola. Indeed, $y = x^2$ is a solution of the equation, so the points on the parabola have the parabola **and the tangent ray** as solutions to the right of the point, but only the parabola itself as a solution to the left. These curves, that follow the parabola to the left of (c, c^2) and the tangent line at that point to the right, are continuously differentiable solutions of the differential equation. All such curves with $c \ge x$ are solutions of the equation passing through (x, x^2) on the curve.

Another example is the set of all lines $y = mx$. These satisfy

$$\frac{dy}{dx} = \frac{y}{x}.$$

The line $x = 0$ must be excluded to have solutions that are functions of x. As in the discussion of (2.2#7), this equation can be turned into the system

$$\frac{dy}{dt} = y$$
$$\frac{dx}{dt} = x$$

that is well-behaved everywhere and has $(0, 0)$ as an **equilibrium point**.

5. Linear homogenous equations.

The general **linear homogenous equation** has the form

$$\frac{dy}{dx} = g(x)y. \qquad (L)$$

This is easily recognized as being separable. There are good reasons for considering such equations as being **special** and to develop **shortcuts** for their solution, but this should not be done at the expense of what these equations say about separable equations in general. In particular, one should resist the use of a formula summarizing the method for solving the equation, because use of the formula requires that the equation be written in a form that exposes the quantities appearing in the formula. A slight change in the interpretation can lead to trying to express the solution in terms of functions that are far from being part of solutions to the equation.

In keeping with this, the solution of equation (L) will be described in a way that avoids producing a formula. When the variables are separated, the equation takes the form

$$\frac{dy}{y} = g(x)\,dx. \tag{LS}$$

The integral of the left side is $\ln y$, and all that we will say about the right side is that **it must be integrated**. There is nothing to be gained by writing the result with an integral sign. A useful expression for the answer can only be found if we succeed in performing the integral, and this becomes harder if the integral is buried inside a formula.

After obtaining the general solution in the form $\ln y = G(x) + C$, where $G(x)$ is any function with $G'(x) = g(x)$, the next step is solve explicitly for y using the fact that $y = e^{\ln y}$. If we had $G(x) = \ln H(x)$ already, then $y = H(x)$ would be seen as a solution of the equation without first writing it as $e^{\ln H(x)}$. This is probably the biggest failing of formulas — they **fail to recognize obvious simplifications**. If the rules for simplifying expressions have been thoroughly mastered, there is no loss in writing the answer in a cumbersome form, but evidence suggests serious misconceptions about this process can come to the surface if they are given a chance. The best protection against this is to introduce new **methods** for solving problems that avoid writing formula that must be simplified.

The **most important** property of linear equations is dependence of the solution on the "$+C$" that appears when the right side is integrated. The step leading to the explicit solution calls for finding

$$e^{G(x)+C} = e^{G(x)}e^{C}.$$

Since the exponential of the **arbitrary constant** C is just **another arbitrary constant**, we have the important

Special Linearity Principle. *If you have one nonzero solution of a first order linear homogeneous equation, then the general solution consists of all multiples of that solution.*

There is a **General Linearity Principle** that is needed for dealing with inhomogeneous equations, higher order equations, or systems, but this already shows why linear equations are special. Instead of relying on correct use of the laws of exponents to process the constant of integration after separating the variables, a **single solution** can be found and the constant of integration inserted **in the right place** afterwards.

6. Exact equations.

Writing equations in terms of **differentials** instead of **derivatives** is also useful for equations that are, or can be easily made, **exact**. These are equations of the form

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

where the **line integral**

$$\int P(x,y)\,dx + Q(x,y)\,dy$$

is **independent of path**. In this case, the integral from some **base point** (x_0, y_0) to the point (x, y) defines a function $F(x, y)$. If the path is taken to the **broken-line** path that consists of the **horizontal segment** from

(x_0, y_0) to (x, y_0) followed by the **vertical segment** from (x, y_0) to (x, y), then $F(x, y)$ is computed by an operation of **partial integration** that is inverse to the operation of partial differentiation. On the first leg of the path $dy = 0$ and one forms

$$\int_{x_0}^{x} P(u, y_0)\, du.$$

The second part of the path adds

$$\int_{y_0}^{y} Q(x, v)\, dv$$

where x is considered as a constant. The first part of the path contributes a function of x alone and the second part gives a function whose partial derivative with respect to y is $Q(x, y)$. The base point can be hidden by finding **any function** whose partial derivative with respect to y is $Q(x, y)$. Then, $F(x, y)$ differs from this by a function of x alone. If the line integral is independent of path, consideration of another broken line path that ends with a segment parallel to the x-axis shows that the partial derivative of $F(x, y)$ with respect to x must be $P(x, y)$. If you really believe the equation to be exact, you find the partial integral of $Q(x, y)$ with respect to y, add the unknown function $g(x)$, differentiate with respect to x and set the result equal to $P(x, y)$. This gives an equation that can be solved algebraically for $g'(x)$. If the equation was exact, the resulting expression depends **only** on x. Integrating gives $g(x)$ and hence $F(x, y)$ up to an additive constant. If the equation was **not** exact, the resulting expression contains y. This tells you to **stop and find a different way to solve the equation**.

It is only necessary to find one function $F(x, y)$ since the general solution of the differential equation is $F(x, y) = C$. Again, the implicit form is preferred unless there is a good reason for finding y as an explicit function of x.

For example, consider this exercise from the textbook:

$$\big(y/x + 6x\big)\, dx + \big(\ln x - 2\big)\, dy = 0 \qquad (x > 0) \tag{2.6\#10}$$

Integrating the expression multiplying dy with respect to y gives

$$F(x, y) = \big(\ln x - 2\big)y + g(x)$$

Differentiating with respect to x and equating to the expression multiplying dx in the equation, and solving, gives

$$\begin{aligned} \frac{y}{x} + g'(x) &= \frac{y}{x} + 6x \\ g'(x) &= 6x \\ g(x) &= 3x^2 \end{aligned}$$

A "$+C$" need not be written at this point since we need only one function $F(x, y)$ and the solution of the equation will be $F(x, y) = C$. In this case, we get

$$\big(\ln x - 2\big)y + 3x^2 = C$$

An explicit solution is found by solving for y to obtain

$$y = \frac{C - 3x^2}{\ln x - 2}$$

One needs $x > 0$ to allow $\ln x$ to be computed, but it is also necessary to exclude $x = e^2$ to avoid dividing by zero. This is known before solving the equation since these values also give trouble in the expression for dy/dx. If y were to be taken as the independent variable, the curve $y = -6x^2$ must be excluded because the slope field is horizontal there. There is no explicit solution for x as an **elementary function** of y, except for constant function $x = e^2$ which now appears as a solution, corresponding to $C = 3e^4$.

Here is an example where the equation fails to be exact

$$(2x + 4y)\,dx + (2x - 2y)\,dy = 0 \qquad (2.6\#2)$$

If we try the same approach, we get

$$F(x, y) = \int 2x - 2y\,dy = 2xy - y^2 + g(x)$$
$$2y + g'(x) = 2x + 4y$$
$$g'(x) = 2x + 2y$$

Since y appears in what should be $g'(x)$, the equation isn't exact and another method must be used to solve the equation. Special methods for such equations are mentioned in Chapter 2, but the method of creating a system can also be used. To use this method, begin by rewriting the equation as

$$\frac{dy}{dx} = -\frac{2x + 4y}{2x - 2y} = \frac{2x + 4y}{2y - 2x}$$

and then separate numerator and denominator to get the linear system

$$\frac{dx}{dt} = 2y - 2x$$
$$\frac{dy}{dt} = 2x + 4y$$

Some care is needed to match the different parts of the equations with the derivatives with respect ot the new variable t. One should avoid trying to remember a formula for this process. Instead, the equation should be solved for dy/dx first. This **allows simplification before moving on to the next step**.

7. Integrating factors.

An equation of the form $P(x, y)\,dx + Q(x, y)\,dy = 0$ is a representative of a family of equations

$$\mu(x, y)P(x, y)\,dx + \mu(x, y)Q(x, y)\,dy = 0$$

for arbitrary functions $\mu(x, y)$. Even if the original equation isn't exact, one of these might be. For example, if a **separable** equation is put into this form with $Q(x, y) = -1$ by multiplying the derivative dy/dx by dx and moving the dy term to the other side of the equation, there will be $\mu(y)$ — a function of y alone — such that multiplying by $\mu(y)$ **separates the variables** The resulting equation, now of the form $P(x)\,dx + Q(y)\,dy = 0$ is exact.

This suggests that we concentrate on the case in which the **integrating factor** μ depends on only one of the variables. The other case is similar, so we describe only the case in which μ is a function of x alone. To find the integrating factor in this special case, we rely on the fact that exactness means that $P(x, y)\,dx + Q(x, y)\,dy = dF(x, y)$ for some function $F(x, y)$. Then, $P(x, y)$ is the partial derivative of

$F(x, y)$ with respect to x and $Q(x, y)$ is the partial derivative of $F(x, y)$ with respect to y. The **equality of mixed partial derivatives** requires that

$$\frac{\partial}{\partial y} P(x, y) = \frac{\partial}{\partial x} Q(x, y).$$

If this criterion is applied to the product with $\mu(x)$, we get

$$\frac{\partial}{\partial y}\big(\mu(x) P(x, y)\big) = \frac{\partial}{\partial x}\big(\mu(x) Q(x, y)\big).$$

On the left side, $\mu(x)$ acts like a constant, but the product rule must be used on the right side. After differentiating, there are two types of terms: $\mu(x)$ times a partial derivative of one of $P(x, y)$ or $Q(x, y)$, or $\mu'(x)$ times $Q(x, y)$. When the terms are collected, the equation must lead to something of the form $\mu'(x) = K(x)\mu(x)$ for some function of x alone, that we give the name $K(x)$. If this happens, we go on to the next step; otherwise, we abandon this approach (unless we are told that it is known to work) and try something different. If there is a strong reason to believe that there is such an integrating factor, but it hasn't been found, either a mistake has been made, or something (like a common factor in the numerator and denominator of a fraction) has been overlooked. When the expected form has been found, it is a linear equation, so it is easily solved. We only need one integrating factor, only one solution is needed. Since nonzero constant multiples of integrating factor are also integrating factors, the **Special Linearity Principle** shows further solutions of the equation determining the integrating factor given nothing new.

After multiplying by the integrating factor, the equation is exact and can be solved by any method known to work for exact equations. One should still be alert, since an error made in finding the integrating factor will lead to an equation that is not exact. Failure of one of the special methods serves to identify a mistake in the determination of the integrating factor. Such mistakes are usually minor and easily corrected, as long as you notice them.

In exercise 22 of Section 2.6 of the textbook, an integrating factor of the form $\mu(x)$ is given. Use the method described here to show how that factor can be found. Similarly, the appearance of y in the denominator of exercise 20 looks artificial. If it is eliminated first, to give

$$\big(\sin y - 2ye^{-x}\sin x\big)\,dx + \big(\cos y + 2e^{-x}\cos x\big)\,dy = 0,$$

there is an integrating factor that depends only on x. Use the method described here to find it. The exact equation that you obtain should agree with the one found by multiplying the equation given in the textbook by the integrating factor given there. Also, **solve exercise 25** by finding an integrating factor depending only on x.

Before computers were taught all the tricks and given a program for trying them, it was common to learn the art of solving differential equations through a collection of special methods. A particular method may not work for an equation, but the way that it would fail might give a clue that could be used to find a better method. Computers have changed this. Symbolic packages like Maple can find closed-form solutions using an extensive collection of tricks; it is also possible to obtain numerical solutions easily, so that the special equations with closed form solutions are less important. The present role of these tricks in the study of differential equations is to provide examples that can be used to illustrate the principles that assure us that differential equations provide useful models of physical or biological systems.

End of Mathematics 244 Summary 1