Mathematics 244: Lab 1

Fall 2003

0. Introduction and Setup. In this lab we use Maple to find exact solutions of differential equations and initial value problems whether the solution is an explicit function or only defined implicitly.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove from the worksheet any extraneous material and any errors you have made.

Before you start up Maple, first copy the seed file for Lab1 into your directory from the Web page of the course. After you start up Maple, import this file into Maple by choosing **Open** from the **File** menu as described in the handout **Instructions for Use of Maple in Mathematics 244**. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own. The ease of obtaining the seed file means that verbatim quotes of Maple instructions in the descriptions can be phased out, so not all prepared instructions will be described here.

The file begins with a preliminary section containing the instructions

```
restart;
with(plots): with(DEtools):
```

as in Lab 0.

1. The first equation. The following commands (already entered into the seed files) show how Maple can find a general solution of the first order equation

$$\frac{dy}{dx} + 2y = \frac{1}{1+e^x}, \tag{1}$$

and how it can be made to evaluate this solution at $x = 0$, and to determine a particular solution with $y(0) = c$. In this problem, the solution y will be found explicitly as a function of x.

```
de1:= diff(y(x),x) + \parama*y(x) = 1/(1 + exp(x));
s1:=dsolve(de1);
y0:=eval(rhs(s1a),x=0);
c1:=solve(y0=c,_C1);
s1a:=eval(s1,_C1=c1);
```

The use of the `rhs` (for "Right Hand Side") function in the definition of `y0` assumes that the solution has been obtained as an **equation** with `y(x)` on the left side. Maple promises to give solutions in this form wherever possible. The quantity `_C1` is a parameter appearing in the general solution. (All recent versions of Maple give the solution in the form expected by these instructions. If this doesn't work, examine the solution to see how `y(x)` is given and what parameter is used.)

1a. Direct solution of initial value problems. This solution above is the same procedure you would use to solve the initial value problem if you were not using Maple. When you **are** using Maple, you can simply execute the following command to solve the initial value problem more simply and directly (note the use of braces { }).

```
s1b:=dsolve({de1,y(0)=c},y(x));
```

Execute this instruction and compare to the previous result. There should be no difference.

The solution produced by Maple is really a family of solutions, i.e., there is a different solution corresponding to each particular choice of the constant c.

1b. Graphing solutions.

The seed file contains statements to find the solutions `t1, t2, t3, t4, t5` corresponding to the choices $c = -2, -1, 0, 1, 2$, and construct a plot of those solutions on the same set of axes. The plot is not shown immediately, but saved to be shown as part of a composite display. Note that when naming the output of a plot, changing the ending semicolon to a colon will suppress output that shows details of the plot structure.

The second component of the display is a **direction field** constructed by the `DEplot` instruction (the complete instruction for this example appears in the seed file). The qualitative behavior of the differential equation can often be seen in such direction fields, although some practice is needed to interpret them. In this lab, direction fields will be used to check results found by other means.

Again, the instruction producing the named plot of the direction field ends with a colon to suppress output. The two plots are combined, and **displayed**, by the `display` command (part of the `plots` library). The instruction in the seed file includes a title. Every plot that appears in your report should have a title.

1c. Discussion.

TThe plot suggests some features of the equation and its solutions that should be discussed. In particular, the picture suggests that, no matter what initial condition we start with, all solutions tend to the same value as $x \to \infty$. **What value is this?** If k is any number smaller than this limiting value, **what does the differential equation say** about the value of dy/dt at a point where a solution to the equation crosses the line $y = k$? **What does the differential equation say** if k is greater than the limiting value?

You should also compare the results obtained by Maple with the solution that the textbook encourages you to use. The Maple instruction `odeadvisor(de1);` tells you how maple classifies the equation in order to select a solution method. Does the answer that Maple gives to this instruction agree with the description of that type of equation used in the textbook? Does the solution appear in the form expected from the method of solution proposed by the textbook for this type of equation?

In the remaining problems, an equation and some initial conditions will be given. Following the pattern of problem 1, **you should divide each problem into three parts**: (a) **construction** of the solutions; (b) **plots** of the solutions and a direction field; and (c) **discussion** of aspects of the solution — especially comparing the Maple solution (as revealed by `odeadvisor`) with methods presented in the textbook. The step-by-step method used initially should not be used for the other equations.

2. The second equation.

Introduce the name `de2` for the equation

$$x\frac{dy}{dx} + xy = 2 - y. \tag{2}$$

Use Maple to find the **solution of the initial value problem** consisting of equation (2) and the initial condition $y(1) = 1/2$ (only one this time). Plot this solution and a slope field over the interval $0 \le x \le 5$ with y restricted to $-2 \le y \le 1$.

Warning: The equation has been written in the notation used in the textbook. This must be translated into the Maple language. In addition to the strict requirements on algebraic expressions, Maple requires that a differential equation be described in terms of $y(x)$ consistently, you cannot abbreviate it to y. Use the solution of Problem 1 as a guide.

As in problem 1, the restriction on y only needs to be given for the slope field. It will be used for the `display`. (If you plot the solution alone with only the restriction on x, the negative values of y of large absolute value as $x \to 0$ will distort the picture. Current versions of Maple are not troubled by these large values. If you are using an old version, you may need further modifications to get a plot.)

Also compare the report from `odeadvisor(de2);` with the form in which the solutions are given.

3. The third equation.

Introduce the name `de3` for the equation

$$\frac{dy}{dx} = \frac{x - e^{-x}}{2y + e^{y}}. \tag{3}$$

and use `dsolve` to solve the equation. (The instructions are in the seed file.) Note in this case, the solution $y(x)$ is only defined implicitly as a function of x, i.e., it appears in an equation involving both y and x in which y is not solved explicitly as a function of x.

If the solution were given in the special form $f(x, y) = \textit{constant}$, the `lhs` function would extract $f(x, y)$ to use as input to the `contourplot` command. currently, the expression `sol3` is not given in this special form, so you should examine it to find the constant introduced into the solution — it is likely to be called `_C1` — and solve for it. An expression that accomplishes this is

```
Val3C:=solve(sol3,_C1);
```

This identifies an expression that is constant on solutions of the equation, so the graph is constructed using

```
g3c:=contourplot(Val3C,x=-2..4,y=-2..4):
```

Consider the initial condition $y(-1) = 0$. The following sequence of commands determine the value of the constant `_C1` corresponding to the initial condition $y(0) = 1$, name the result `c`, and then replace the arbitrary constant `_C1` in the solution by this value, and, finally, use the `implicitplot` command is used to construct a graph of the solution of the initial value problem

```
c:=simplify(eval(Val3C,{x=-1,y(x)=0}));
sol3p:=subs(_C1=c,sol3);
g3p:=implicitplot(sol3p,x=-2..4,y=-2..4,color=BLUE):
```

The implicit solution `sol3p` should have a form equivalent to

$$(y(x))^2 + e^{y(x)} = \frac{1}{2}x^2 + e^{-x} + \frac{1}{2} - e.$$

To find the value of the solution $y(x)$ at say $x = 3$, we substitute $x = 3$ in the above equation and then determine the number y which satisfies the resulting equation

$$y^2 + e^{y} = 5 + e^{-3} - e.$$

Maple's `fsolve` command can solve the above equation to obtain the value $y(3)$. The seed file contains instructions to accomplish this. (These instructions were tested on Maple7 and Maple9. Neither accepts `y(3)` as the name of a variable in `fsolve`, but allows it to be replaced by an acceptable name using `subs`. Feel free to use an alternative solution — as long as it leads to a numerical answer. As with previous equations, compare the result of `odeadvisor` with the classification in the textbook.

4. The fourth equation.

Consider the **logistic equation**

$$\frac{dy}{dt} = \frac{4}{3} y \left(1 - \frac{y}{2}\right). \tag{4}$$

As in previous exercises, use the instructions `dsolve`, `DEplot`, and `odeadvisor` to find the solution, plot a slope field for $-2 \le t \le 2$ and $-1 \le y \le 3$, and discover how Maple classified the equation.

Use appropriate instructions to obtain solutions with initial conditions $y(0) = 1/2$, and $3/2$. Plot graphs of these on the interval $-2 \le t \le 2$ and combine this graph with the slope field you obtained.

Describe connections between the report of `odeadvisor`, the form of the solution given by `dsolve`, and the method used in the text for solving this equation.

5. The fifth equation.

Consider the equation

$$y + \left(2x - ye^{y}\right)\frac{dy}{dx} = 0, \tag{5}$$

which is essentially the equation of Exercise 21 of Section 2.6. As in previous exercises, introduce `de5` as a name for the equation. Then, use the instructions `dsolve`, `DEplot`, and `odeadvisor` to find the solution, plot a slope field for $-2 \le t \le 2$ and $-1 \le y \le 3$, and discover how Maple classified the equation.

The textbook informs you that this equation has an **integrating factor** $\mu(x, y) = y$. Maple knows this, too! The instruction `iF:=intfactor(de5);` will assign this integrating factor to the name `iF`. Then, use the instruction `de5a:=iF*de5;` to create a new equation that has been multiplied by this integrating factor. Describe any changes in the result of applying `dsolve` and `odeadvisor` to this equation instead of the original `de5`.