Mathematics 244: Lab 3

Fall 2003

0. Introduction and Setup.

In this lab, we use Maple to examine differential equations modeling forced and unforced oscillations. In particular, we shall consider linear and nonlinear models of a pendulum and also explore the phenomena of beats and resonance.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove from the worksheet any extraneous material and any errors you have made.

Before you start up Maple, first copy the seed file into your directory from the Web page of the course. After you start up Maple, import this file into Maple by choosing **Open** from the **File** menu as described in the handout **Instructions for Use of Maple in Mathematics 244**. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own.

1. The linear model.

Consider the motion of a pendulum, which consists of a mass m attached to one end of a rigid rod of length L. The other end of the rod is fixed at a point O and the rod is free to rotate about O. The position of the pendulum at time t is described by the angle $\theta(t)$ between the rod and the downward vertical direction, with the counterclockwise direction taken as positive (see Figure 9.2.2 on page 474 of the text). The differential equations governing the motion of the pendulum are

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

It is expected that small changes in the equation will lead to small changes in the solution. Two modifications designed to approximate the equation by one that is more easily solved are to ignore damping by setting $c = 0$ or to replace $\sin\theta$ by θ to get a linear equation (or both). Note that θ is just the first term of the Taylor series for $\sin\theta$ about $\theta = 0$. The solutions to this linear equation should give a good approximation to those of the general equation (G). In a mathematical treatment, only the values of the coefficients and not their expression in terms of physical measurements will enter, so we write the general equation as

$$\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} + K\sin\theta = 0 \qquad (G)$$

and its linear approximation as

$$\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} + K\theta = 0 \qquad (L)$$

This study is expected to take place on earth, where the constant g is the acceleration due to gravity will be $9.81\text{m}/\text{sec}^2$. If we fix the length L at 1.09 meters, then $K = g/L = 9$. The examples that we consider will have values of the measurements m and c so that $\mu = c/(mL)$ takes on the values 3, 6, and 8. We also fix the initial conditions $\theta(0) = 0.5$, $\theta'(0) = 4.5$ corresponding to the pendulum being pushed vigorously and released one radian from its rest position. These general definitions are collected in the following statements that appear at the start of the seed file.

```
K:=9;
deG:=diff(theta(t),t,t) + mu*diff(theta(t),t)+K*sin(theta(t))= 0;
deL:=diff(theta(t),t,t) + mu*diff(theta(t),t)+K*theta(t)= 0;
Iv:=theta(0)=0.5, D(theta)(0)=4.5;
```

1a. Exact solution. The `dsolve` command can be used to compute the exact solution of the linear models. It's use is not exactly the same as for first order equations, since we now have two initial conditions. To solve equation (L) without damping and plot the solution, execute the following commands.

```
soln1a:=dsolve({subs(mu=0,deL),Iv});
gr1a:=plot(rhs(soln1a),t=0..10,color=BLUE):
display(gr1a,title="Undamped linear model");
```

Note: this has been tested with Maple7 and later versions, but may not work with earlier versions. If only one function appears in the equation, it is currently necessary to give the `dsolve` function only one argument which is a **set** consisting of the equation and initial conditions. If there is a typographical error in describing the initial value problem, the response is likely to be a cryptic error message.

Also note that there is one statement where the **plot structure** `gr1a` is constructed without being shown (you don't **ever** want to see a plot structure), including a choice of color for the graph. This is followed by a `display` command which provides a title and **draws** the plot.

1b. Damped motion. Redefine μ using the other values (3, 6, and 8) and repeat the `dsolve` instruction used in part (a) to obtain solutions to the linear damped equations for the three cases $\mu = 3, 6, 8$ and construct plots of the solutions over the interval $0 \le t \le 10$, assigning appropriate names to the results. As in part a, introduce names for plot structures and use the `display` command to produce **titled** graphs of these solutions.

Using the form of the equation and the appearance of the graph, **discuss** how these examples illustrate **underdamping**, **critical damping**, and **overdamping**. Plot the solutions for $\mu = 0$ and $\mu = 3$ on the same graph and use this to identify effects of a small amount of damping. Similarly, plot the solutions for $\mu = 6$ and $\mu = 8$ on the same graph and use this to identify effects of overdamping.

2a. Graphical solutions. For the nonlinear equations, an **exact** solution in **elementary functions** cannot always be found. In such cases, the standard use of the `dsolve` command will not give a useful result. For this equation, different releases of Maple have given different results. With Maple9, a general solution can be found, but there is no attempt to use it to solve initial value problems.

The `DEplot` command works for higher order equations, but it shows only a graph (there is no direction field since it is a **second derivative** that is found from the equation and solutions are determined by giving both a **point** and a **direction**). This command uses the classical fourth order Runge-Kutta method, so it will often be necessary to set a step size to get an accurate graph. We first illustrate something that doesn't quite work,

The seed file contains the following instructions to illustrate the limitations described above and, finally, produce a usable plot. The plot will be given a name for later use, and then displayed by entering that name as a command.

```
dsolve({deG,Iv});
DEplot(subs(mu=0,deG),theta(t),t=0..10,{[Iv]},title="Undamped
nonlinear model with default step size");
opt2:=linecolor=PLUM,stepsize=0.25;
gr2:=DEplot(subs(mu=0,deG),theta(t),t=0..10,{[Iv]},opt2):
display({gr1a,gr2},title="Comparison of undamped models");
```

Discuss features of the graph with the default step size that indicate that it is likely to be inaccurate. Also **discuss** the following topics suggested by the comparison of the general undamped model with its linear approximation: the exact solution of the linear equation is known to be periodic, does the nonlinear model appear to have the same property? Compare amplitude of the solutions to the two equations. Compare intervals between peak values for the two models.

2b. Numerical solutions. Another approach is to use the `dsolve` command with the `numeric` option. This uses an **adaptive** (i.e., self-correcting) method for solving the equation, making the result available in a form that allows Maple to either **evaluate** or **plot** the solution. In many cases, this will produce accurate solutions without the human assistance required for the classical method used in 2a.

To explore this method of solution and its uses, execute the following commands.

```
soln2a:=dsolve({subs(mu=0,deG),Iv},numeric,range=0..10);
soln2a(1.2101);
soln2a(0.5513);
gr2a:=odeplot(soln2a,color=TAN);
display({gr2a,gr2},title="Comparison of numerical methods");
```

Discuss the nature of information returned by the `soln2a` function created by `dsolve` in this example, and significance of the particular values $t = 0.5513$ and $t = 1.2101$ used as illustrations.

2c. Comparison. Use the `numeric` option of `dsolve` to obtain solutions to the nonlinear damped equations for the case $\mu = 3$ and compare these values to those given by the exact solution of the linear version of this equation at $t = 1, 2, 3, 4, 5$.

Compare these results to those shown in a plot the solutions over the interval $0 \le t \le 10$.

End of Lab 3