Mathematics 244 Essay 1

Hanging Cables

Fall 2004

0. Introduction and Setup.

The function $\cosh x$ appeared in one of the examples done in the first lecture. As an aside, it was mentioned that the graph of $y = \cosh x$, known as a **catenary**, arises as the shape of a **hanging cable**. The proof of this is a nice illustration of the ability of differential equations to model physical systems, so it should be included in the course.

The ability to produce this essay so quickly is a consequence of being able to follow the treatment in the Calculus text of *Thomas & Finney*, although we do not follow it very closely. (I used the eighth edition, *Addison-Wesley*,1992, but the example probably appears in all editions.)

We are told that a cable is supported at two points and hanging freely between those points, and we want to find an equation of the curve giving the shape of the cable.

1. The variables.

All quantities in this analysis are functions of x which represents the horizontal distance along the line directly under the cable. Several of those quantities will be named in order to formulate equations (both direct relations between the variables and differential equations) that can be solved to determine the shape of the cable. The variables that we use (in addition to x) are:

- y: the height of the curve above the ground, so that the expression for y as a function of x is the solution to the problem.
- $y' = \dfrac{dy}{dx}$: the derivative of y with respect to x.
- θ: the angle that the tangent line at (x, y) makes with the ground, so that $y' = \tan\theta$.
- T: the internal force (tension) of the cable at (x, y).

  There is also a physical parameter:
- δ: the (weight) density of the cable.

Weight density instead of mass density is used to allow the use of only one parameter.

It is assumed that the distances x and y are measured in the same units, which need not be specified. The quantities y' and θ are **dimensionless**. T is the only force, but the units for measuring forces appear implicitly in the density δ. A change of units for distances or forces will only introduce scale factors into the values of the variables and the parameter δ. This will be seen to have a minor effect on the analysis used to derive and solve the equation describing this system.

2. Physical assumptions.

Each piece of the cable is subject to the force of gravity, pointing **straight down** and **proportional to the length** of the cable and the **tension forces** at the ends of the cable pointing in the **tangential directions** at those points. Thus the **force vector** due to the tension of the cable at (x, y) is $\pm\langle T\cos\theta, T\sin\theta\rangle$ with the sign chosen to point outward on the interval describing the cable segment. The cable is **assumed to be at rest**, so the sum of these force vectors must be zero.

3. A small piece of cable.

These conditions must hold for **all** segments of the cable, do we can consider the derivatives of all functional relationships between our variables as well as the relationships themselves. In fact, those derivatives will suffice since recovering the relationships themselves is just a matter of **solving a system of differential equation**. The general solution of those equations will introduce arbitrary constants to allow initial conditions to be satisfied. This turns out to have a simplifying effect because it hides the effect of the **arbitrary choices** of origin and scale of the xy-coordinates as well as the deferring the need to specify the points at which the cable is supported.

The effect of these derivatives can be described informally in the language of differentials. As long as you talk about differentials in a way that can be justified by differentiating relations that can be derived on the macroscopic scale, the results will be valid. The use of differentials only serves to provide a more intuitive description of the limit process.

We consider a piece of cable at the point (x, y) whose projection on the ground has length dx. This piece has length $\sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx$. The downward force is thus $\delta\sqrt{1 + y'^2}\,dx$. The other force is the difference of the tension forces at the ends of the piece of cable. This is

$$d\langle T\cos\theta, T\sin\theta\rangle = T\langle -\sin\theta, \cos\theta\rangle d\theta + \langle\cos\theta, \sin\theta\rangle dT,$$

which must balance the weight.

4. The horizontal component.

In particular, the horizontal component of this force must be zero. This is expressed by the separable differential equation

$$\cos\theta\, dT = T\sin\theta\, d\theta,$$

which can be solved as follows:

$$\begin{aligned}
\frac{dT}{T} &= \frac{\sin\theta\, d\theta}{\cos\theta}\\
\int\frac{dT}{T} &= \int\frac{\sin\theta\, d\theta}{\cos\theta}\\
\ln T &= -\ln\cos\theta + C\\
\ln T &= \ln\sec\theta + C\\
T &= K\sec\theta
\end{aligned}$$

with $K = e^C$. This gives $dT = K\sec\theta\tan\theta\, d\theta$.

5. The vertical component.

Now, balance the vertical components.

$$\delta\sqrt{1 + y'^2}\,dx = T\cos\theta\, d\theta + \sin\theta\, dT$$

$$\delta\sqrt{1 + y'^2}\,dx = K\sec\theta\cos\theta\, d\theta + K\sin\theta\sec\theta\tan\theta\, d\theta$$

$$\delta\sqrt{1 + y'^2}\,dx = K\left(1 + \tan^2\theta\right)d\theta$$

$$\delta\sqrt{1 + y'^2}\,dx = K\sec^2\theta\, d\theta = K\, d\left(\tan\theta\right)$$

$$\delta\sqrt{1+y'^2}\,dx = K\,dy'$$

$$\int dx = \int \frac{K\,dy'}{\delta\sqrt{1+y'^2}}$$

Note that this relates x with the slope y'. We haven't gotten to anything that involves y itself. In Calc II, this integral was evaluated by the substitution $y' = \tan\theta$. This would eventually lead to the desired equation, but it is easier in this case to use the substitution $y' = \sinh u$. This leads to

$$dy' = \cosh u\,du$$
$$\sqrt{1+y'^2} = \cosh u$$

Thus, $dx = (K/\delta)\,du$. This reveals that $(K/\delta)u = x - C_0$ for some constant C_0. Thus, $y' = \sinh\big((\delta/K)(x - C_0)\big)$. Integrating this gives

$$y - C_1 = \frac{K}{\delta}\cosh\left(\frac{\delta}{K}(x - C_0)\right).$$

As predicted, the quantities K, C_0, and C_1 that were introduced as **constants of integration** correspond to the location of the origin and choice of units of measurement.

These constants started life as the $+C$ of ordinary integrals, but those integrals were embedded in functions, so their contents appeared throughout the final formula. However, those locations turned out to have a natural interpretation when related to the underlying physical problem.

6. Interpreting initial conditions. The definition in terms of e^x shows that $\cosh x$ is an **even function**. Thus, if our cable is suspended between two points with the same value of y, C_0 is the midpoint of the x coordinates of those points. This also gives the minimum point on the curve. Such symmetry is expected, so it is reassuring that it appears so easily. If you are given the height of the cable at the endpoints and at the minimum, the values of K and C_1 are found by solving simple equations.

If the heights at the two ends are not equal, solving for the parameters is a little more difficult, but the analysis of the special case of endpoints of equal height shows that these conditions lead easily to expressions for K and C_1 in terms of C_0. Varying the force on the cable will cause its shape to vary through the family of catenaries with different values of C_0.

7. Suspension bridges. A similar analysis applies to finding the shape of the support cable of a suspensions bridge. Many features are the same as in the preceding study. The difference in this case is that the weight of the cable is negligible, but it is subject to a downward force caused by the (heavy) weight of the bridge that it is supporting. If the bridge roadway is level, then the weight of a small segment is proportional to dx. The same variables can be used, and analysis of the horizontal component is the same, so we still have $T = K\sec\theta$ and $dT = K\sec\theta\tan\theta\,d\theta$ for some constant K. However, the left side in our analysis of the vertical component is replaced by the simpler expression $\delta\,dx$. The last step of that derivation is replaced by

$$\int \delta\,dx = \int K\,dy'.$$

Thus, $y' = (\delta/K)(x - C_0)$ and

$$y - C_1 = \frac{\delta}{2K}(x - C_0)^2.$$

The shape of this cable is a parabola. Initial conditions can be treated similarly and the algebra needed to identify the solution with given initial conditions will be simpler.

8. An important principle.

This shows that **physical similarities** often lead to **similar differential equations**, but **the solutions may show no relation** to one another.

In particular, the only similarity between the parabola and the catenary is that they each have an axis of symmetry and their defining functions have a second derivative of constant sign. All quantitative properties of the curves are very different.

9. The Gateway Arch.

Yet another example deals with designing a monument in which the weight of a large arch would be supported entirely on its base.

Photo source: http://www.stlouisarch.com/

Stability of the structure requires that there be no horizontal forces on pieces of the arch. Since only the weight of the arch itself is involved, the analysis is the same as that of the heavy cable. Reversing the direction of the force vectors has only a minor effect on the mathematics. The arch has the shape of a catenary. More information about the Gateway Arch can be found at http://www.stlouisarch.com/ and http://nps.gov/jeff/.

End of Mathematics 244 Essay 1