Mathematics 244 Essay 4

Series Methods

Fall 2004

Introduction

The fourth segment of the course works through Chapter 5 of the text on **Series Solutions of Differential Equations**. All equations dealt with in the text are **linear** and **homogeneous** and most have coefficients that are polynomials in the independent variable (which is usually x here). The examples are all **second order**. If they were first order, there would be the temptation to get a closed-form solution by the methods of Chapter 1. Higher order equations can be solved using the same method, although the computation would be more tedious. Indeed, most equations that we consider here have a special form that only appears in second order equations in which there is one special solution that only contains even powers of the independent variable and another that only contains odd powers.

Many of these equations have been found to have useful applications and are named after someone who determined interesting properties of the equation or its solutions.

Although **only** linear homogeneous equations are studied in this course, series solutions of nonlinear equations can be found in a similar way, and the solution in the linear case may become clearer if it is extended to an **undetermined coefficients** method for nonhomogeneous equations.

Differential operators

The textbook introduces names for the coefficients of the equation. Equation (1) of section 5.2 is

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

to **declare** that x is the **independent variable**, $P(x)$, $Q(x)$, and $R(x)$ are **given** functions of x (usually polynomials), and that y is the **dependent** variable. A **solution** of the equation is an expression giving y as a function of x.

For some purposes, it is convenient to have **fixed names** for the coefficients of the equation. In particular, if there were a formula for the solution in terms of the coefficients, the formula would use this notation, and the process of solving the equation would consist of **identifying** $P(x)$, $Q(x)$, and $R(x)$. The main step in the solution would be to manipulate a given equation **algebraically** to reveal these functions.

The method proposed here will be **less rigid**. It will only be required that the equation be tested for being **linear** and **homogeneous**. This means that the equation must be written as a sum of **terms**, each of which is a function of x multiplied by **exactly one** factor that is a derivative of y to some order. **Zero order differentiation**, giving y itself, is allowed, and almost always present. In particular, Bessel's equation (treated in detail in section 5.8) could be met as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = \nu^2y,$$

with constant ν, and the only algebra required would be to rewrite it as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y - \nu^2y = 0.$$

Only zero survives on the right side, and the **four** terms on the left side have the required form. It is not necessary to **collect** terms having the same order of differentiation. The notation used in this essay will be to introduce the name $L[y]$ for the left side of the equation, and to refer to it as a **differential operator**. A **homogeneous equation** we are trying to solve is then **always** $L[y] = 0$. This convention has been used informally elsewhere in the course. In particular, when nonhomogeneous equations were considered, they were often referred to as $L[y] = g(x)$ in order to have a way to describe the related homogeneous equation as $L[y] = 0$.

You know that $y(x)$ is a solution of an equation $L[y] = 0$ if the **computation** of $L[y(x)]$ simplifies to zero. Therefore, a **key step** in any solution must be the **evaluation** and **simplification** of $L[y(x)]$ for given **test functions** $y(x)$. This idea was exploited in **undetermined coefficient** methods for solving nonhomogeneous equations. When solving an equation in series, the test functions are just the pure powers x^k. For **linear equations**, the effect of the differential operator L on any polynomial can be found once the effect on powers is known. The textbook accomplishes this by introducing names for the coefficients of the polynomial and writing an expression for the action of strongeach term of $L[y]$ on the sum. The expression is only simplified **after** it is written in this form. However, it is **much easier** to simplify $L[x^k]$ for individual k and to introduce the coefficients **after** performing these simplifications. No "shift of the index of summation" is required, because **there is no index of summation**. The **recurrence formula** for the coefficients of $y(x)$ that is the goal of this approach is only sought after a few instances of the formula have been obtained. If the goal is only to obtain a few terms of the series, the general recurrence formula need **never** be found. The equations determining the coefficients of the series are found **one at a time**, and the relevance of the method of solution is always **visible**. There is no mysterious formula to memorize.

Ordinary points

If the equation satisfies the hypothesis of the Existence and Uniqueness Theorem at a point, that point is called **ordinary**. Other points are called **singular**. For second order linear equations whose coefficients are polynomials, the only singular points are the roots of the coefficient of d^2y/dx^2 (called the **leading coefficient** of the equation).

In almost all examples, our solutions will be assumed to be of the form

$$y = \sum_{k=0}^{\infty} a_k x^k. \tag{1}$$

Such **power series** are known to converge in sets of the form

$$\{\, x : |x| < r \,\}$$

that are symmetric intervals around $x = 0$. It is useful to allow **complex** values of x, and this property persists to obtain a **circle** of convergence in the complex plane. We make no direct use of this, but we present the following **fact**: the series solution (1) of a differential equation **converges in the largest circle around the origin that does not include a singular point of the equation**. In particular, if the leading coefficient is a constant, all points are ordinary and the series has an infinite radius of convergence, which requires that $|a_k|$ approaches zero **rapidly** — the series for e^x is a typical example.

If it is ever necessary to consider series in powers of $(x - x_0)$ to describe solutions with initial conditions at points $x = x_0$ with $x_0 \neq 0$, it will usually be best to make an **explicit** change of variable $u = x - x_0$ **in the equation**. Since $dx/du = 1$, we have $dy/du = dy/dx$, so this process is only an algebraic change of variables on the coefficients. The equations that we consider are simple enough that this change of variable is easier than including a **base point** in our work with power series. Indeed, after changing the variable, we may recognize an equation that we have already solved.

If the leading coefficient is $1 + x^2$, there are no real singular points, but $x = \pm i$ are singular. The series (1) will have radius of convergence 1. If the series obtained for the solution with **particular initial**

conditions at $x = 0$ is used to find the value of the solution and its derivative at $x = 1/2$ (which is within the interval of convergence), those values could be used to find a series represent **the same** solution as a series in powers of $(x - 1/2)$. The distance from $1/2$ to $\pm i$ is $\sqrt{5}/2$, so this series will converge as far as $(1 + \sqrt{5})/2$. The initial conditions at $x = {}^1\!/_2$ can be found from those at $x = 0$ by using the original series. This is an example of **analytic continuation** producing series on **overlapping intervals**. Repeated use of this technique can extend the original solution to all real x.

Initial conditions

For second order equations at an ordinary point, there will be a unique solution to the equation for any **initial conditions** giving values of $y(0)$ and $y'(0)$. Substituting $x = 0$ into the series (1) tells us that $a_0 = y(0)$. Differentiating (1) term-by-term and substituting $x = 0$ into the result tells us that $a_1 = y'(0)$. (While the validity of these operations requires **proof**, the results are correct for the functions that we meet in this course, allowing series to be manipulated exactly like **polynomials with infinitely many terms**.) To solve the equation, one simply substitutes the series (1) into the equation and collects all terms of the same degree in the result. The coefficients of the resulting series **must all be zero** for a solution of a homogeneous equations. This gives a sequence of linear equations in the a_k. We will see that the constant term of the result involves only a_0, a_1, and a_2; the linear term involves these together with a_3; and, in general, the coefficient of x^m involves a_k with $0 \le k \le m + 2$. In particular, each **one** new term introduces **one** new a_k. This allows all a_k to be found in terms of a_0 and a_1. The connection between the a_k and the standard form of initial conditions tells us that this is the right way to organize the process of solving infinitely many equations. Wherever we stop, we will have all the terms of the series through some degree, and the conventional use of series treats these polynomials as a sequence of approximations to the function represented by the series.

In particular, the exam problems in this course will ask for a small number of initial terms of a series solution, so they can be found by finding a **polynomial** P for which $L[P]$ has no terms of low degree.

In many cases, the sequence of equations can be described by a single equation involving a parameter representing the degree of the term in $L[y]$ that can be solved to get a recurrence formula for the a_j. We do not follow this direction in most examples, preferring to concentrate on the obtaining the **numerical** values of the initial terms in the series.

Exploiting linearity

Instead of forming one complicated expression involving the a_k, we have proposed organizing the work around computing expressions $L[y]$, that are equivalent to substituting only the $y = x^k$ into the equation. For example, consider the equation in Exercise 9 of Section 5.2:

$$(1 + x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0.$$

Here,

$$L[y] = \frac{d^2y}{dx^2} + x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y$$

Here are the first few values of $L[x^k]$:

$$\begin{aligned}
L[1] &= 6 \\
L[x] &= \quad 2x \\
L[x^2] &= 2 \\
L[x^3] &= \quad 6x \\
L[x^4] &= \qquad 12x^2 \quad + \quad 2x^4 \\
L[x^5] &= \qquad\qquad 20x^3 \quad + \quad 6x^5
\end{aligned}$$

We have arranged these results with terms of the same degree arranged in columns. Linearity gives the equation

$$L\Big[\sum a_k x_k\Big] = \sum a_k L[x_k]$$

which tells us to multiply the **rows** by $a_0, a_1, a_2, a_3, a_4, a_5$ and add the **columns** to get the equations in the a_k that say that the series (1) satisfies the differential equation. In this case, those equations are

$$\begin{array}{rcrl} 2a_2 & + & 6a_0 & = 0 \\ 6a_3 & + & 2a_1 & = 0 \\ 12a_4 & & & = 0 \\ 20a_5 & & & = 0 \end{array}$$

In each of these rows, the first term is a multiple of the a_k, starting with a_2, that will be found by solving that equation, and the other terms have smaller subscripts. For this equation, we find that $a_2 = -3a_0$, $a_3 = -a_1/3$, but $a_5 = a_4 = 0$. Additional equations will show that all subsequent $a_k = 0$. All solutions of this equation are polynomials, since the general solution is

$$y = a_0\big(1 - 3x^2\big) + a_1\big(x - \frac{x^3}{3}\big).$$

Although we only expected a series with a radius of convergence of 1, we found polynomial solutions. The points $x = \pm i$ are still singular, since Abel's theorem tells us that the **Wronskian** of a basis for the solutions will be a constant multiple of $(1 + x^2)^2$, which you could verify directly for this basis. The singularity is due to all solutions having the same value of $y'(x)/y(x)$ at $x = i$ or $x = -i$.

Parameterizing the solutions in terms of a_0 and a_1 allows the solution satisfying given initial conditions and $x = 0$ to be written immediately.

Solutions will rarely be this simple, but we should organize work so that **such good fortune doesn't become an obstacle** to getting the correct answer.

If we needed additional rows in this table, they could be derived from the general formula

$$L[x^k] = (k^2 - 5k + 6)x^k + (k^2 - k)x^{k-2}$$

Thus, there are at most two terms, but the second term is zero if $k = 0$ or $k = 1$ and the first term is zero if $k = 2$ or $k = 3$ (these special cases were noted in the discussion above). The process of **adding the columns** involves identifying x^m as appearing only when $m = k$ or $m = k - 2$. The total coefficient of x^m is

$$\big((m+2)^2 - (m+2)\big)a_{m+2} + \big(m^2 - 5m + 6\big)a_m.$$

Setting these expressions equal to zero gives the **recurrence formula**

$$a_{m+2} = \frac{(m-2)(m-3)}{(m+2)(m+1)}a_m$$

that computes the a_k in order. In this case, the recurrence tells us that $a_4 = a_5 = 0$ and then all successive $a_m = 0$. In this form, it **fails to apply** when $m = -1$ or $m = -2$, corresponding to the fact that a_0 and a_1 are **allowed to be nonzero** while a_m for $m < 0$ must be zero. This **always happens** when building a series solution at an **ordinary point**, and a variant on this theme will appear in the **method of Frobenius** for finding solutions at **regular singular points** that will appear later in this essay.

A useful family of equations

This example is a special case of the equation

$$(a+bx^2)\frac{d^2y}{dx^2}+cx\frac{dy}{dx}+(r+sx^2)y=0. \tag{2}$$

with constants a, b, c, r, s (the gap in naming the coefficients results from avoiding letters that are not usually used as names for coefficients since they have fixed meaning elsewhere). The point $x=0$ is an ordinary point if $a \neq 0$.

Again, the left side will be denoted $L[y]$ and we find

$$L[x^k]=ak(k-1)x^{k-2}+\big(bk(k-1)+ck+r\,\big)x^k+sx^{k+2}$$

You should not treat this as a **formula** — it merely echoes the terms that appear when you compute $L[x^k]$. It is given here to show the result of the calculation that is a **direct translation** of the symbol $L[x^k]$. This allows a **theoretical discussion** of this family of equations. In particular, the formula reminds you that a term x^m appears in such an expression only if $m=k-2$, $m=k$ or $m=k+2$. To obtain the **recurrence formula**, these conditions are solved for k in terms of m to give $k=m+2$, $k=m$, and $k=m-2$, respectively, and the terms in x^m in these $L[x^k]$ combined (in the examples, k indexes the **rows** and m indexes the **columns**). When $a \neq 0$, the x^{k-2} has a zero coefficient precisely when $k=0$ or $k=1$, for which $k-2$ is negative. There is an additional simplification when $s=0$ removing the x^{k+2} term. If $s=0$, the recurrence formula gives a_m as an explicit multiple of a_{m-2} for all $m \geq 2$. Otherwise, there is a **three term recursion** that gives a_{m+2} in terms of a_m and a_{m-2} (the latter term is assumed to be zero when $m=0$ or $m=1$) for $m \geq 0$. **In all case**, the values of a_0 and a_1 are **found from the initial conditions**.

The Hermite equation

Exercise 21 of Section 5.2 deals with the equation

$$y''-2xy'+\lambda y=0$$

which is known as the **Hermite equation**. Here, λ is a constant. The solutions of the equation for various λ will be described. Denoting the left side of this equation by $L[y]$, we find

$$L[x^k]=k(k-1)x^{k-2}+(\lambda-2k)x^k.$$

In particular,

$$\begin{array}{rcccccccccccccc}
L[1] & = & \lambda \\
L[x] & = & & & (\lambda-2)x \\
L[x^2] & = & 2 & + & & & (\lambda-4)x^2 \\
L[x^3] & = & & & 6x & + & & & (\lambda-6)x^3 \\
L[x^4] & = & & & & & 12x^2 & + & & & (\lambda-8)x^4 \\
L[x^5] & = & & & & & & & 20x^3 & + & & & (\lambda-10)x^5
\end{array}$$

Multiplying row k by a_k and adding gives the following results as the sums of the columns

$$2a_2+\lambda a_0=0$$

$$6a_3+(\lambda-2)a_1=0$$

$$12a_4+(\lambda-4)a_2=0$$

$$20a_5+(\lambda-6)a_3=0$$

which are instances of the recurrence formula $m(m-1)a_m + (\lambda - 2m + 4)a_{m-2} = 0$, that asserts that the coefficient of x^{m-2} in $L[y]$ is zero. In **solved form**, this says

$$a_m = \frac{2m - 4 - \lambda}{m(m-1)} a_{m-2}.$$

Again, we see a denominator of $m(m-1)$ corresponding to the fact that a_0 and a_1 are determined from initial conditions while $a_m = 0$ for $m < 0$. The **solved form of the recurrence** is only used for $m > 1$, but this restriction is possible only because the principal form of the recurrence allows arbitrary values of a_0 and a_1 while terms whose index is not a nonnegative integer are zero. The numerator shows that **one solution** will be a polynomial when λ is a positive even integer. For example, when $\lambda = 6$, one has $a_5 = 0$ which leads to $a_m = 0$ for all larger odd numbers. The solution with $a_0 = 0$ and $a_1 = 1$ is $x - (2/3)x^3$, while the solution with $a_0 = 1$ and $a_1 = 0$ is the infinite series

$$1 - 3x^2 + \frac{1}{2}x^4 + \frac{1}{30}x^6 + \frac{1}{280}x^8 + \frac{1}{2520}x^{10} + \dots$$

The Legendre equation

Exercises 22 through 29 of Section 5.3 deal with the equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

which is known as the **Legendre equation**. Here, α is a constant. Later, we will specialize α to an integer, but it is convenient to write $\beta = \alpha(\alpha + 1)$ while describing the series solution of the equation. Denoting the left side of this equation by $L[y]$, we find

$$L[x^k] = k(k-1)x^{k-2} + (\beta - k^2 - k)x^k.$$

In particular,

$$\begin{array}{rclllllllllll}
L[1] & = & \beta \\
L[x] & = & & & (\beta - 2)x \\
L[x^2] & = & 2 & & + & & (\beta - 6)x^2 \\
L[x^3] & = & & & 6x & & + & & (\beta - 12)x^3 \\
L[x^4] & = & & & & & 12x^2 & & + & & (\beta - 20)x^4 \\
L[x^5] & = & & & & & & & 20x^3 & & + & & (\beta - 30)x^5
\end{array}$$

Multiplying row k by a_k and adding gives the following results as the sums of the columns

$$2a_2 + (\beta)a_0 = 0$$

$$6a_3 + (\beta - 2)a_1 = 0$$

$$12a_4 + (\beta - 6)a_2 = 0$$

$$20a_5 + (\beta - 12)a_3 = 0$$

which are instances of the recurrence formula $m(m-1)a_m + (\alpha - m + 2)(\alpha + m - 1)a_{m-2} = 0$, that asserts that the coefficient of x^{m-2} in $L[y]$ is zero. In **solved form**, this says

$$a_m = \frac{(m - 1 + \alpha)(m - 2 - \alpha)}{m(m-1)} a_{m-2}.$$

As in the case of the Hermite equation, a_0 and a_1 are determined from initial conditions while $a_m = 0$ for $m < 0$, and the solved form of the recurrence is only used for $m > 1$. When α is an integer, the recurrence shows that some coefficient is zero, and all larger coefficients of the same parity are also zero. In particular, $a_m = 0$ for $m = \alpha + 2$ and $m = 1 - \alpha$. Only integer values of m greater than 1 appear in this recursive definition of a_m, so exactly one of these is relevant for any integer value of α. In particular, if $\alpha \geq 0$, $a_{m+2} = 0$. Since the recurrence relates terms with indices differing by 2, one of the solutions is a polynomial of degree α. If α is even, the solution with $y(0) = 1$ and $y'(0) = 0$ is such a polynomial; if α is odd, the solution with $y(0) = 0$ and $y'(0) = 1$ is such a polynomial. The other solution is easily described as a series. For example, when $\alpha = 4$, the equation

$$(1 - x^2)y'' - 2xy' + 20y = 0$$

has one solution that is the polynomial

$$1 - 10x^2 + \frac{35}{3}x^4$$

and another solution that is a series

$$x - 3x^3 + \frac{6}{5}x^5 + \frac{2}{7}x^7 + \frac{1}{7}x^9 + \frac{1}{11}x^{11} + \frac{28}{429}x^{13} + \dots$$

The general theory predicts that these series are sure to converge for $|x| < 1$. The polynomial solution is defined everywhere, giving an example showing that is **sometimes** possible to get solutions valid beyond the guaranteed interval. The other solution has the ratio of consecutive terms given by an expression whose limit is x^2. The **ratio test** proves that the series converges for $|x| < 1$. We will return to this function later to study its behavior near $x = 1$.

Euler equations

If $a = s = 0$ in (2), $L[x^k] = \big(bk(k-1) + ck + r\big)x^k$, so $y = x^k$ is a solution if $bk(k-1) + ck + r = 0$. This is a quadratic equation in k which usually has two roots. When there are two distinct roots, we have two solutions whose Wronskian is different from zero everywhere except at the singular point $x = 0$.

Since $x = 0$ is a singular point, solutions should be found separately for $x > 0$ and $x < 0$. As noted in the textbook, the solutions $y = x^k$ are easily interpreted when $x > 0$, and the corresponding functions of $-x$ should be used when $x < 0$. The interpretation is based on writing $x^k = e^{k \ln x}$. The similarity with the method of solution in the constant coefficient case is explained by the change of variables $x = e^t$. This is outlined in Exercise 23 of Section 5.5.

This change of variables tells us how to handle the cases where $bk(k-1) + ck + r = 0$ does **not** have two distinct real roots. Note that **it is not necessary to be able to deduce the solution**. Once you have **reason to believe** that a particular function satisfies the equations, **it is only necessary to verify that it is a solution**. The theory assures us that linear combinations of two solutions that are linearly independent (except at the singular point) give **all** solutions of the equation.

A pair of conjugate complex roots $k = \alpha \pm \beta i$ lead to complex exponential functions that can be expressed in terms of the real functions $C(x) = x^\alpha \cos(\beta \ln x)$ and $S(x) = x^\alpha \sin(\beta \ln x)$. It is a **useful exercise** to show that

$$xC'(x) = \alpha C(x) - \beta S(x)$$

$$xS'(x) = \beta C(x) + \alpha C(x)$$

and to use this to show that $C(x)$ and $S(x)$ satisfy the equation

$$x^2y'' + (1 - 2\alpha)y' + (\alpha^2 + \beta^2)y = 0,$$

which is an equation that leads to the exponentials that we rewrote as linear combinations of $C(x)$ and $S(x)$. The calculation is aided by using the product rule to show that

$$x\frac{d}{dx}\big(xf'(x)\big) = x\big(xf''(x) + f'(x)\big) = x^2 f''(x) + xf'(x).$$

For equations with constant coefficients, the case of **repeated roots** called for multiplying the exponential solution by t to get an independent solution. The change of variables suggests that, for Euler equations, the solutions should be x^k and $x^k \ln x$. Again, it is easily shown that both of these functions satisfy

$$x^2y'' + (1-2k)y' + k^2y = 0,$$

which is the equation that leads us to an algebraic with k as a repeated root.

Regular Singular points

If an equation **can be written** as the sum of an Euler equation and some higher order terms, it is said to have a **regular singular point** at $x = 0$. There are series solutions in this case, but the lowest degree term of the series must be one of the solutions of the Euler equation part of the equation. Our special form (2) is a second order equation with a regular singular point if $a = 0$ and $b \neq 0$.

If both $a = 0$ and $r = 0$ in (2), all terms have a factor of x. Removing that factor leads to the **equivalent** equation

$$bx\frac{d^2y}{dx^2} + c\frac{dy}{dx} + sxy = 0.$$

It should not be necessary to multiply this equation by x before looking for a solution. A better approach is to find $L[x^k]$ in **all** cases, and to call a singular point “irregular” if it is still **not** possible to use this to find a solution.

To illustrate the method, consider Problem 1 of Section 5.6:

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$$

This does not appear to be in the form (2), but acquires that form if multiplied by x. The difference in detail are minor enough that we work with the equation in the form in which it was given.

As usual, the left side of the equation will be denoted $L[y]$. The main technique if to find $L[x^k]$:

$$L[x^k] = 2k(k-1)x^{k-1} + kx^{k-1} + x^{k+1} = (2k^2-k)x^{k-1} + x^{k+1} = k(2k-1)x^{k-1} + x^{k+1}.$$

Constant multiples of x^m appear in $L[x^k]$ if $m = k+1$ or $m = k-1$. That is, if $k = m-1$ or $k = m+1$. If $k = m-1$, the term is x^m; if $k = m+1$, the term is $(m+1)(2m+1)x^m$. To have a solution that is a sum of terms a_kx^k, then the a_k must satisfy the **recurrence**

$$a_{m-1} + (m+1)(2m+1)a_{m+1} = 0.$$

Note that this notation is **different from that used in the text**. We use indices that **agree with the exponent** on x without first factoring out the lowest degree term. From this recurrence we see that it is possible for $a_{m+1} \neq 0$ while $a_{m-1} = 0$ if $(m+1)(2m+1) = 0$. This **indicial equation** that selects the possible lowest degree terms $m+1 = 0$ or $m+1 = 1/2$ of series solutions is part of the general method for solving the equation and **not** a separate **pre-processing** step. In this notation, the recurrence will give the value of

$a_2, a_4, a_6 \ldots$ as multiples of a_0 and $a_{5/2}, a_{9/2}, a_{13/2}, \ldots$ as multiples of $a_{1/2}$. Taking one of these leading coefficients to be 1 and the other 0 gives solutions

$$y_1 = 1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \frac{1}{11088}x^6 + \ldots$$

and

$$y_2 = x^{1/2} - \frac{1}{10}x^{5/2} + \frac{1}{360}x^{9/2} - \frac{1}{28080}x^{13/2} + \ldots$$

The coefficients of both series can be **proved** to be fractions with numerator 1 and **rapidly increasing** denominators. Apart from the factor of $x^{1/2}$ in one of these solutions, these series give functions that have Taylor series that **converge everywhere**. This is the main significance of **regular** singular points: they allow series solutions to be centered at them that will converge at least as far as the nearest **other** singular point. At the same time, the leading term of the series describes the behavior of the solution **near** the point since **all other terms** of the series will be much smaller than the leading term.

Nonhomogeneous equations

The text limits itself to homogeneous equations in this chapter, but similar methods apply to nonhomogeneous equations. The broader view may help to clarify the role of the exponents that appear in the solutions.

A **differential operator** $L[y]$ typically expresses $L[x^k]$ as a polynomial (or series) in which the lowest power of x that appears is x^{k-n} for some n. In particular, $n = 2$ when $x = 0$ is an ordinary point, $n = 0$ for the usual equation with a regular singular point, and $n = 1$ for the equation considered in the previous section. However, there are special values of k for which the coefficient of this term is zero. For ordinary points, these values are **always** $k = 0$ and $k = 1$. For regular singular points, they are the solutions of the indicial equation. For these values of k, we do not have an expression ϕ for which $L[\phi]$ has k^{k-n} as its lowest degree term, although we will find such an expression soon.

If we want to solve the nonhomogeneous equation $L[y] = \alpha(x)$ where $\alpha(x)$ is given as a sum of constant multiples of powers of x, we can find ϕ such that $L[\phi]$ has lowest degree term equal to the term of lowest degree in $\alpha(x)$. This allows our equation to be written

$$L[y - \phi] = \alpha(x) - L[\phi]$$

where the right side now has its **lowest degree term** of **higher degree** than the lowest degree term in $\alpha(x)$. If we **never meet** one of our special exponents, this process determines a series solution of the nonhomogeneous equation. The steps in this process always increase the index by an integer. In particular, if the indicial equation has complex roots, all integer powers behave normally, so there is no difficulty finding a power series for one **particular solution** of $L[y] = \alpha(x)$ when $\alpha(x)$ is given by a power series, although the general solution will be complicated.

The special values of k have the property that $L[x^k]$ has only terms of higher degree than expected. Except in special cases, this means that $\alpha(x) = -L[x^k]$ is an expression for which the nonhomogeneous equation has a series solution ϕ. In this case, $x^k + \phi$ is a solution of the **homogeneous** equation $L[y] = 0$. This is the same solution that was obtained using the recurrence formula. These special $L[x^k]$ cause a disturbance in the sequence of exponents of lowest degree terms: some values are skipped, and others are duplicated. The duplicated values allow two different solutions to some nonhomogeneous equations, whose difference is a solution of the homogeneous equation. This happens at most twice for a second order equation.

This can fail to find a solution to the homogeneous equation if the duplicated degree of one $L[x^k]$ meets a skipped degree while constructing the series. This only happens if the skipped degree exceeds the duplicated degree by an integer. A repeated root of the equation determining the special degrees also fails to find a second solution of the homogeneous equation.

If $y_1(x)$ is a solution of the homogeneous equation $L[y] = 0$, then skipped degrees can be filled in by considering $\phi = y_1(x) \ln x$, since

$$\frac{d}{dx}\big(y_1(x) \ln x \big) = y_1'(x) \ln(x) + \frac{y_1(x)}{x}$$

and

$$\left(\frac{d}{dx}\right)^2 \big(y_1(x) \ln x \big) = y_1''(x) \ln(x) + 2\frac{y_1'(x)}{x} - \frac{y_1(x)}{x^2}.$$

From this, it follows that the terms containing $\ln x$ disappear from $L[y_1(x) \ln x]$ and the remaining terms give a term of the **expected** lowest degree in $L[y_1]$. As an example, consider Exercise 3 of Section 5.6, which has $L[y] = xy'' + y$. Then,

$$L[x^k] = k(k-1)x^{k-1} + x^k,$$

whose lowest degree term usually has degree $k - 1$. The special values of k are $k = 0$ and $k = 1$ — just like the case of an ordinary point, although $x = 0$ is singular in this case. Here, $L[1] = 1$ and $L[x] = x$. The next case, $L[x^2] = 2x + x^2$, is typical of all other cases. From this, we see that -1 is a missing minimal degree, and 1 is duplicated. The recurrence formula is $a_m + m(m+1)a_{m+1} = 0$, so starting with $a_1 = 1$ leads to

$$\begin{aligned} y_1(x) &= x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k!(k-1)!} \end{aligned}$$

Then,

$$\begin{aligned} y_1'(x) &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{kx^{k-1}}{k!(k-1)!} \\ &= \sum_{n=0}^{\infty} (-1)^{n} \frac{x^n}{(n!)^2} \end{aligned}$$

with $n = k - 1$, and $y_1''(x) = -y_1(x)/x$. Then

$$\begin{aligned} L[y_1(x) \ln x] &= xy_1''(x) \ln(x) + y_1(x) \ln x + 2y_1'(x) - \frac{y_1(x)}{x} \\ &= 2y_1'(x) - \frac{y_1(x)}{x} \\ &= 1 - \frac{3}{2}x + \frac{5}{12}x^2 - \frac{7}{144}x^3 + \dots \end{aligned}$$

This duplicates degree 0, so there is a power series ψ, which **may taken** without a term of degree 1 to avoid retracing the determination of the first solution, such that $L[\psi] = L[y_1(x) \ln x]$. If we take

$y_2(x) = \psi - y_1(x)\ln x$, then $L[y_2] = 0$ and $L[y_2(x)\ln x]$ involves only powers of x with lowest exponent -1. This second solution starts with

$$y_2(x) = 1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \dots$$

The same method may be employed to solve a nonhomogeneous equation $L[y] = \alpha(x)$ where $\alpha(x)$ is given by a power series. The expressions $L[x^k]$ have lowest degree terms of all nonnegative integer degrees (with $L[x]$ and $L[x^2]$ both starting with terms of degree 1). This allows a series $z = \sum c_k x^k$ to be found, with $c_1 = 0$ such that $L[z] = \alpha(x)$. The **general solution** of $L[y] = \alpha(x)$ is then $z + C_1 y_1 + C_2 y_2$. Requiring $c_2 = 0$ was a **normalizing assumption**. Adding a multiple of y_1 gives a solution y_2 with any desired coefficient in the degree one term. This also works if $\alpha(x)$ has a singularity at $x = 0$ causing it to be expressed as a power series multiplied by a **general** power of x. The only difficulty for **this** $L[y]$ occurs when that power is a negative integer since we do not yet have an expression z for which $L[z]$ has lowest degree -1. However, $z = y_2 \ln x$ can be used in that role.

Legendre's equation revisited

Exercise 11 in section 5.6 suggests the change of **independent variable** $x = 1 + t$ in Legendre's equation.

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

to obtain

$$(-2t - t^2)y'' + (-2 - 2t)y' + \alpha(\alpha+1)y = 0.$$

This reinterprets the singular point at $x = 1$ as $t = 0$. Denoting the left side of this equation by $L[y]$, as usual, we find that

$$L[t^k] = -2k^2 t^{k-1} + \big(\alpha(\alpha+1) - k(k+1)\big)t^k.$$

The shift of origin has broken the symmetry that gave degrees differing by 2, but there are still only two terms in the expression for $L[t^k]$ — this time with consecutive degrees. The point $t = 0$ is a **regular singular point** since the lower degree term in $L[t^k]$ has a zero coefficient if $k = 0$. If α is positive integer, the higher degree term is zero for $k = \alpha$, leading to a polynomial solution as before. With $\alpha = 4$, we get

$$\begin{array}{rcrcrcrcr}
L[1] & = & 20 & & & & & & \\
L[t] & = & -2 & + & 18t & & & & \\
L[t^2] & = & & & -8t & + & 14t^2 & & \\
L[t^3] & = & & & & & -18t^2 & + & 8t^3 \\
L[t^4] & = & & & & & & & -32t^3
\end{array}$$

From this, one finds that the solution of the equation that is equal to 1 when $t = 0$ is

$$P = 1 + 10t + \frac{45}{2}t^2 + \frac{35}{2}t^3 + \frac{35}{8}t^4.$$

This polynomial is $^3/_8$ at $t = -1$, so it must be $^3/_8$ times the result of substituting $x = 1 + t$ in our previously obtained polynomial solution $1 - 10x^2 + 35x^4/3$. It is a **useful algebraic exercise** to check this.

The general theory of repeated root (or roots differing by an integer) in the indicial equation of a regular singular point tells us that $L[P\ln t]$ will be a polynomial. Maple gives

$$L[P\ln t] = -41 - 210t - \frac{645}{2}t^2 - \frac{385}{2}t^3 - \frac{315}{8}t^4.$$

Using our previously computed values of $L[t^k]$, we can find a polynomial Q of degree 4 — which may be taken without constant term — so that $L[P\ln t + Q]$ is a constant times t^4. With the aid of Maple, one finds

$$L\left[P\ln t - \frac{41}{2}t - \frac{579}{8}t^2 - \frac{1781}{24}t^3 - \frac{4717}{192}t^4\right] = -\frac{315}{8}t^4$$

Then, using $L[t^k]$ for $k \geq 5$ additional terms in the series can be found. Direct use of Maple's routines for finding series solutions of differential equations verifies the terms found above.

It is not immediately clear **which** solution this gives. When $x = 0$ was our base point, a basis of solution was given by the polynomial P with $P(0) = 1$ and $P'(0) = 0$. A natural choice for the second solution was determined by the initial values $y(0) = 0$ and $y'(0) = 1$. Our new solution is expressed in terms of this basis by finding the values of this function and its derivative at $t = -1$ (which is the **same point** as $x = 0$).

This example requires computer assistance to get accurate answers easily, but equations having a simple polynomial solution allow easy computation of terms of the other solution by this method.

Exercises In each of these exercises, you will be given $L[y]$ with polynomial coefficients. Begin by finding a general formula for $L[x^r]$. The result will be a **finite sum** of coefficients depending on k times powers of x. Find the values of r for which the lowest power of x in the general expression for $L[x^r]$ has a zero coefficient. If $x = 0$ is an ordinary point, these will be $k = 0$ and $k = 1$; if $x = 0$ is a regular singular point, other values of k are possible — even complex values. While there are singular points that are not regular, they won't appear in these exercises. This leading coefficient gives the **indicial equation**; and the special values of r that you found are the **exponents of the singularity** r_1 and r_2. These values are the possible **leading degrees** of series solution. Specialize your formula for $L[x^r]$ to $r = r_i, r_i + 1, r_i + 2$ for $i = 1, 2$ (at most six different values), and use these to find the **first three nonzero terms** of each series solution of the equation (if possible — there may be some examples where only one series can be found). If $r_1 \geq r_2$ and $r_1 - r_2$ is an integer, there will be a solution y_1 corresponding to the exponent r_1, but there may not be a second **series solution**. In this case, the first few terms of $L[y_1 \ln x]$ should be found. Using this term **in place of** $L[x^{r_1}]$ allows a second solution to be found. Again, only three terms need be found. In these exercises, you will not be told which case applies to which equation. Since most of the work consists of finding $L[x^r]$, prior knowledge of a possible singularity at $x = 0$ is not necessary to begin the solution.

1. $L[y] = (1 + x^2)y'' + 4xy' - 10y$.
2. $L[y] = (x^2 + 2x^4)y'' + xy' - y$.
3. $L[y] = 2x^2y'' + 3xy' - y$.
4. $L[y] = (1 - 3x^2)y'' + 2xy' + (5 - 2x^2)y$.
5. $L[y] = xy'' + 2xy' - 4y$.
6. $L[y] = x^2y'' + (5x^2 - 2)y$.

End of Essay 4