

FIRST MIDTERM for MATH 244

October 11, 2004

(20) 1.) Consider the initial value problem

$$y' + \frac{3/2 \cdot x^2 + x}{4y^3} = 0, \quad y(1) = 1$$

- (a) What type of first-order differential equation is this?
- (b) Find the solution to the initial value problem.
- (c) Extra question (5 extra points): Determine the x values for which the solution exists.

Solution:

- (a) This is a separable differential equation as it can be rewritten in the form

$$4y^3 y' = 3/2 \cdot x^2 + x.$$

integrating both sides (w.r.t. x and using chain rule) we obtain

$$y^4 = \frac{x^3}{2} + \frac{x^2}{2} + C.$$

The constant C can be determined by the initial value $y_0 = 1$ at $x_0 = 1$.

$$1^4 = \frac{1^3}{2} + \frac{1^2}{2} + C$$

which yields $C = 0$. Therefore, the implicit solution for the initial value problem is

$$y^4 = \frac{x^3}{2} + \frac{x^2}{2}$$

In order to be able to solve for y we have to guarantee that the l.h.s. is positive

$$\begin{aligned} x^3 + x^2 &\geq 0 \\ x^2(x+1) &\geq 0 \end{aligned}$$

as x^2 is always positive this is equivalent to

$$\begin{aligned}x + 1 &\geq 0 \\x &\geq -1\end{aligned}$$

Under this condition we can take the fourth root

$$y = \pm \left(\frac{x^3}{2} + \frac{x^2}{2} \right)^{1/4}$$

Still we have the ambiguity of the sign. Rechecking the initial value

$$\begin{aligned}1 &= \pm \left(\frac{1^3}{2} + \frac{1^2}{2} \right)^{1/4} \\1 &= \pm 1^{-1/4}.\end{aligned}$$

Therefore, we see that only the $+$ -branch fulfills the initial condition and the solution is

$$y(x) = \left(\frac{x^3}{2} + \frac{x^2}{2} \right)^{1/4} \quad x \geq -1.$$

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(20) 2.) Consider the initial value problem

$$4y + 3x^2 + (y^4 + 4x)y' = 0, \quad y(0) = 1$$

- (a) Is this an exact equation?
- (b) Give the implicit solution for the initial value problem. (If the equation is not exact use an integrating factor of the form $\mu(x)$).

Solution:

- (a) For this equation $M(x, y) = 4y + 3x^2$ and $N(x, y) = y^4 + 4x$. An equation of this form is exact if and only if

$$\frac{\partial}{\partial y}M(x, y) = \frac{\partial}{\partial x}N(x, y).$$

Computing this derivatives we see that

$$\begin{aligned}\frac{\partial}{\partial y}M(x, y) &= 4 \\ \frac{\partial}{\partial x}N(x, y) &= 4\end{aligned}$$

the equation is exact. We find the implicit solution by constructing a function $\psi(x, y)$ such that

$$\begin{aligned}\frac{\partial}{\partial x}\psi(x, y) &= 4y + 3x^2 \\ \frac{\partial}{\partial y}\psi(x, y) &= y^4 + 4x.\end{aligned}$$

Integrating the first equation w.r.t. x (considering y as a parameter) one obtains

$$\psi(x, y) = \int \frac{\partial}{\partial x}\psi(x, y)dx = \int 4y + 3x^2 dx = 4yx + x^3 + C(y)$$

for an unknown constant $C(y)$ which depends on the parameter y . This function can be determined using the second equation:

$$\frac{\partial}{\partial y}(4yx + x^3 + C(y)) = 4x + C'(y)$$

should be equal to

$$\frac{\partial}{\partial y}\psi(x, y) = y^4 + 4x.$$

To guarantee this, we have to choose $C(y)$ such that $C'(y) = y^4$ or

$$C(y) = \int y^4 dy = \frac{y^5}{5} + C_0.$$

The arbitrary constant we can drop and obtain as a solution for

$$\psi(x, y) = 4yx + x^3 + \frac{y^5}{5}.$$

This means that we can rewrite our initial differential equation as

$$\frac{\partial}{\partial x}\psi(x, y(x)) + \frac{\partial}{\partial y}\psi(x, y(x))y'(x) = 0$$

which is the same as

$$\frac{d}{dx}(\psi(x, y(x))) = 0.$$

This yields that the implicit solution is

$$C = \psi(x, y) = 4yx + x^3 + \frac{y^5}{5}.$$

for an arbitrary constant C which has to be chosen according to the initial condition $y(0) = 1$

$$C = 4 \cdot 1 \cdot 0 + 0^3 + \frac{1^5}{5} = \frac{1}{5}.$$

Finally, we showed that the implicit solution to the initial value problem is

$$\frac{1}{5} = 4yx + x^3 + \frac{y^5}{5}.$$

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(20) 3.) Consider the initial value problem

$$ty' - 3y = t^5 e^t, \quad y(1) = 1$$

- (a) What type of differential equation is this?
- (b) What is the largest interval on which the solution exists according to the general theory. (Do not use the solution to answer this question).

- (c) Find the solution.
- (d) Extra question (5 extra points): Explain what happens if one try to solve the initial value problem $y(0) = 1$.

Solution:

- (a) Dividing both sides by t we can write the differential equation as

$$y' - \frac{3}{t}y = t^4 e^t$$

which is a linear differential equation with non-constant coefficients $p(t) = -3/t$ and $g(t) = t^4 e^t$.

- (b) The function $g(t)$ is continuous for all t . The function $p(t)$ is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$. As $1 \in (0, \infty)$ the solution exists for sure on this interval.
- (c) In order to solve the differential equation we have first to determine the integrating factor $\mu(t)$. We want a function $\mu(t)$ such that

$$(\mu(t)y(t))' = \mu(t) \left(y'(t) - \frac{3}{t}y(t) \right)$$

or if we calculate the l.h.s. using product rule

$$\mu'(t) = -\frac{3}{t}\mu(t).$$

This is a separable equation and hence can be solved as follows

$$\begin{aligned} \int \frac{1}{\mu} d\mu &= \int -\frac{3}{t} dt \\ \ln |\mu| &= -3 \ln |t| + C_0 \end{aligned}$$

where C_0 is an unknown constant. Solving for μ we get (using $e^{a+b} = e^a \cdot e^b$) that

$$|\mu| = e^{-3 \ln |t| + C_0} = e^{C_0} e^{-3 \ln |t|} = e^{C_0} |t|^{-3},$$

where we used $a^{bc} = (a^b)^c$. Hence we obtain dropping the arbitrary factor $\pm e^{C_0}$

$$\mu(t) = t^{-3}.$$

Using this integrating factor we can rewrite our initial differential equation as

$$(t^{-3}y(t))' = te^t.$$

This is easy to solve by integrating both sides w.r.t. to t

$$t^{-3}y(t) = \int se^s ds = te^t - e^t + C,$$

for an unknown integration constant C . Solving for y we obtain

$$y(t) = t^4 e^t - t^3 e^t + Ct^3.$$

The constant C can be determined by the initial value

$$1 = y(1) = 1^4 \cdot e - 1^3 \cdot e + C \cdot 1^3 = C.$$

Therefore, the solution to the initial value problem is

$$y(t) = t^4 e^t - t^3 e^t + t^3.$$

(d) To this problem there is no solution as

$$y(0) = 0^4 \cdot 1 - 0^3 \cdot 1 + C \cdot 0^3 = 0.$$

This can happen because $p(t)$ is not continuous at $t = 0$ and therefore the existence result is not applicable here.

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(20) 4.) A home buyer can afford to spend no more than \$800 per month on mortgage payments. Suppose that the annual interest rate is 9% and that the term of the mortgage is 20 years.

- (a) Give a model for this situation assuming that the interest is compound continuously and that the payment are also made continuously. (Hint: compute time in years and show that the interest rate for continuous compound is $r = \ln(1.1) \approx 0.1$)
- (b) Solve the corresponding differential equation.

Solution:

- (a) Denote by $S(t)$ the part of the mortgage in dollar which is still to be paid. Let t be in years. Under the assumption of continuous compound the interest rate causes an increase of

$$r \cdot S(t)$$

and the payments a decrease of

$$-800 \cdot 12$$

Together, this gives the differential equation

$$S'(t) = rS(t) - 800 \cdot 12$$

r is the interest rate for continuous compound, whereas 10% is the interest rate for annual compound. We have first to find the appropriated r reflecting this annual rate. Let $\tilde{S}(t)$ be the amount to pay for mortgage, if there had been no payments. On the one hand, the corresponding differential equation is

$$\tilde{S}'(t) = r\tilde{S}(t)$$

with solution

$$\tilde{S}(t) = S_0 e^{rt}.$$

On the other hand, we know that for an annual rate of 10% the solutions is

$$\tilde{S}(t) = S_0(1 + 0.1)^t.$$

Matching this to results we can determine r

$$\begin{aligned} e^{rt} &= (1.1)^t \\ rt &= \ln(1.1)t \\ r &= \ln(1.1) \approx 0.1 \end{aligned}$$

This yields the following differential equation for S

$$S'(t) = 0.1 \cdot S(t) - 9600$$

(b) Solving this linear differential equation with constant coefficients we obtain

$$S(t) = S_0 e^{0.1 \cdot t} + (1 - e^{0.1 \cdot t}) 9600/0.1$$

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(20) **5.)** Give an approximative solution for $y(1)$ where $y(t)$ is the solution of the following initial value problem

$$y'(t) = t + \frac{2}{y}, \quad y(0) = 1$$

using Euler's method for step size $h = 1$ and step size $h = 0.5$. It is not necessary to simplify the results. Euler's method gives a sequence of approximate values y_n by the following iteration formula: $t_n = n \cdot h$, h step size, (t_0, y_0) initial condition

$$y_n = y_{n-1} + f(t_{n-1}, y_{n-1}) \cdot h$$

Solution: The function f is in this case $f(t, y) = t + \frac{2}{y}$. The initial value gives us $y_0 = 1$, $t_0 = 0$.

Step size $h = 1$: As the step size is 1 and y_n is an approximation for $y(t_n)$, we have to find n such that $1 = t_n = nh = n$. So we just have to compute y_1 . The iteration formula in this case is given by

$$y_1 = y_0 + f(t_0, y_0) \cdot h.$$

Thus we get

$$y_1 = 1 + f(0, 1) \cdot 1 = 1 + (0 + 2/1) \cdot 1 = 3.$$

Step size 0.5: Again y_n is the approximation for $y(t_n)$. In this case we need n such that $1 = t_n = n \cdot 0.5$. This means $n = 2$. So we have to compute y_2 . Since h changed we have to recompute y_1

$$y_1 = y_0 + f(t_0, y_0)h = 1 + f(0, 1) \cdot 0.5 = 1 + (0 + 2/1) \cdot 0.5 = 2.$$

To calculate y_2 we have to apply the iteration formula again for $t_1 = h = 0.5$

$$\begin{aligned} y_2 &= y_1 + f(t_1, y_1) \cdot h = 2 + f(0.5, 2) \cdot 0.5 = 2 + (0.5 + 2/2) \cdot 0.5 \\ &= 11/4 = 2.75 \end{aligned}$$

The correct result is $y(1) = 2.67$.

